New Results on the DBM Modelling, Forecasting and Control of Global Climate data

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Abstract

This report considers an inductive, Data-Based Mechanistic (DBM) approach to modelling the dynamic relationship, in the form of an ordinary differential equation model, between global averaged measures of total radiative forcing and atmospheric temperature, in which the model structure is inferred from the data. Noting the dominant modal behaviour of large climate models, it shows that the information content in these data alone can only support a low order model between the total forcing and temperature but that the residuals of this model can be modelled as a stochastic quasi-cyclic signal with a central period of approximately 52 years. The complete stochastic model identified in this manner is then used to produce rolling 10-year-ahead forecasts over the historical data including a prediction of the leveling out of temperature over the last ten years, which derives in part from the effects of the quasi-periodic component. This component has a fairly strong correlation with the Atlantic Multi-decadal Oscillation (AMO) index and this leads to a two input, Hypothetico Inductive DBM (HI-DBM) model that explains 92.5% of the changes in the the Global temperature anomaly and can be used for forecasting and control studies, as well as providing a basis for further HI-DBM modeling.

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1 Introduction

Most environmental modelling research conforms to the ‘scientific method’, following what Popper (1959) has termed a ‘hypothetico-deductive’ method. But Popper’s philosophical stance is based mainly on his consideration of laboratory-based science: one only has to search for the word ‘experiment’ in his famous book to realise how important experimentation, and particularly planned experimentation, is important in his view of the scientific method. As he says:

Only when certain events recur in accordance with rules or regularities, as is the case with repeatable experiments, can our observations be tested . . . We do not take even our own observations quite seriously, or accept them as scientific observations, until we have repeated and tested them. Only by such repetitions can we convince ourselves that we are not dealing with a mere isolated coincidence, but with events which, on account of their regularity and reproducibility, are in principle inter-subjectively testable.

Unfortunately, it is rarely possible to conduct well planned experiments on natural systems in order to remove the often inherent ambiguity in the observations and so clarify our understanding of the potentially complex mechanisms that underlie this observed behaviour. The climate scientist, for instance, can hypothesize about the causes and mechanisms of ‘global warming’ but it is clear from the controversy on this topic, that such hypotheses, although widely recognized in the scientific community, are not universally accepted. And this is clearly a situation where planned experimentation is impossible.

In the case of global climate systems, the difficulties associated with hypothetico-deductive modelling are exacerbated by their perceived complexity. For instance, global climate models, such as the Atmosphere-Ocean General Circulation Models (AOGCMs)) discussed in chapter 9 of the 2013 International Panel on Climate Change (IPCC) report (see https://www.ipcc.ch/report/ar5/wg1/), are very large and complex simulation models. The present report will confirm previous research (see e.g. Young et al., 1996; Young and Jarvis, Young and Jarvis) that, despite this complexity, the globally averaged response characteristics of these models, even when they are ‘emulated’ by smaller but still quite high order models, are defined almost completely by a relatively small number of ‘dominant modes’; and it is these
dominant modes of behaviour that are most important in applications such as forecasting and the management (control) of carbon emissions (see e.g. Young and Jarvis, Young and Jarvis; Young, 2006; Jarvis et al., 2008). Unfortunately, this introduces major difficulties for the estimation of the model parameters because it means that, in the absence of planned experimentation, the information content of the observational data is insufficient to allow for unambiguous verification of the assumed model structure and estimation of the parameters that characterize this structure.

The hypothetico-deductive approach to scientific research can be contrasted with the inductive method, which has a rich history in science and was, indeed, the normal approach to scientific research used during the era of the ‘natural philosopher’, the term used to describe the scientist prior to the twentieth Century (see e.g. Young, 2012). Induction is the opposite of the deduction: it starts by taking as many observations of the natural system as possible, with the aim of inferring from these data how the system works. Of course, in the real world, this differentiation between different approaches to scientific research is too simplistic. Inductive and hypothetico-deductive modelling are synergistic activities, the relative contributions of which will depend upon the system being modelled and the information of all types, not only time-series data, that are available to the scientist and modeller.

In order to emphasize the importance of this synergy, the author has suggested a hypothetico-inductive approach (Young, 2013) to the modelling of observational data from natural systems. In this Data-Based Mechanistic (DBM) method of modelling, the model structure is initially identified from the naturally occurring data, thus avoiding undue reliance on prior hypotheses and ensuring that the resulting models are fully identifiable from the available data. Subsequent to this initial analysis, however, those hypotheses that are consistent with the inductively inferred model structure are considered in order to ascertain which of them are most appropriate for describing the system within the context of the modelling objectives. These modelling objectives are, of course, most important: there is rarely only one model of a real system because different objectives demand different types of model.

As we shall see in subsequent sections, the objectives of the present report are to synthesise a continuous-time differential equation DBM model that is well suited for relatively short term prediction of globally averaged temperatures. As such, the model needs to efficiently represent the dominant modal behaviour of the global climate system; and it has to be a stochastic model,
obtained using optimal methods of statistical inference, so that it can quantify the uncertainty in the predictions over the forecasting interval. Time series and forecasting models of this general type have been the subject of previous research in the climate literature (see e.g. Kaufmann et al., 2006b,a; Mills, 2009, 2010a,b). However, these previous studies are concerned mainly with the identification and use of ‘black-box’, difference equation representations of the climate sampled data, that derive mainly from an econometric approach to time-series analysis. Although the DBM models considered in the present report are developed initially in a black-box transfer function form and utilize methods of statistical identification and estimation that are related to those used in econometrics, the transfer functions are defined in continuous-time and represent differential equations.

Differential equation models have a number of advantages, from both scientific and statistical standpoints (Garnier and Young, 2014). In the context of climate system modelling, for instance, differential equation models are of the same general form as those used in the construction of global climate models; and they are characterized by parameters that have immediate physical meaning and appropriate units that quantify this meaning. Consequently, it is normally possible to interpret these models in physically meaningful terms that adds to their credibility in scientific terms. Moreover, as we shall see in the next section 2, these same models can be used for the emulation of large climate models.

2 DBM Evaluation of Large Climate Models

The main aim of the present report is to develop a stochastic DBM model between total radiative forcing and the Global Temperature Anomaly (GTA), based on the globally averaged climate data plotted in Figure 1 (for description, see later, section 5), it is necessary first to justify the use of such a model in the context of other, more conventional climate models. At first sight, such models are quite different to DBM models: in particular, they are most often of very high dynamic order. For instance, the large Coupled Atmosphere-Ocean GCM (AOGCM) global climate models, (e.g. HadCM3, developed at the Hadley Centre in the United Kingdom; ECHA-M3, developed by the Max Planck Institute for Meteorology; and GFDL15a, developed by the US National Oceanic and Atmospheric Administration (NOAA) at the Geophysical Fluid Dynamics Laboratory), require powerful computers.
Figure 1: Globally averaged climate data: Temperature anomaly, top panel; Total radiative forcing, 2nd panel; CO$_2$ component of total forcing, 3rd panel; and the AMO index in the bottom panel.

for their implementation. Even with such computers, however, such models are characterized by long computational times for their solution and their
internal dynamic mechanisms are both complex and difficult to analyze in simple terms.

For most scientists not directly associated with such large climate models, their basic concepts and mechanisms are better understood by reference to much less complex but still relatively high dynamic order ‘emulation’ models, such as the Model for the Assessment of Greenhouse Gas Induced Climate Change (MAGICC) (see e.g Meinshausen et al., 2011) that has been employed in the preparation of various (IPCC) Assessment Reports. Such emulation models, as well as their large model progenitors, are mainly deterministic models based of the numerical solution of differential and partial differential equations. They are sometimes employed for uncertainty and sensitivity studies using Monte Carlo ensemble analysis but they are not normally identified or estimated using standard statistical methods, such as maximum likelihood, because their large parameterization makes this impossible unless the parameterization is heavily constrained in some way (see below).

2.1 The MAGICC Model

The large climate models and their associated emulators tend to reflect the current views of the climate science community about the model structure and its parameterization. In other words they represent the hypothetical aspects of the hypothetico-deductive scientific method. The parameter values that characterize this hypothesized structure are obtained in various ways but this will not be discussed in detail here. It will suffice, for illustrative and comparative purposes, to consider a version of one specific model: the MAGICC emulation model mentioned above. The MAGICC model is composed of an interconnected set of differential equations and the Matlab version of MAGICC used in the present report was developed by David Leedal and Andrew Jarvis (pers. comm.), who kindly made the program available to the present author: the parameters of this model and the values used are shown in Table 1: the associated model has 40 ocean layers, each with depth 100 metres, and it is solved with a numerical solution step of 0.1 years.

It is not possible to simultaneously estimate all the model parameter values in Table 1 from the data in Figure 1 because there is insufficient information in these data to allow for such statistical estimation. In other words, the model is over-parameterized so some other method of inference must be employed. There are two major alternatives for such inference: a largely deterministic approach, where most parameter values are constrained to those
specified by the climate modeller on the basis of other scientific analysis and measurement, with the remaining parameters optimized using a standard gradient method of optimization; and a numerical Bayesian stochastic approach, using software such as the DREAM algorithm of Vrugt et al. (2009), where normally all the parameters are estimated but with the prior probability distributions constrained so that estimation is based on the immediate locality of the prior. In other words, both approaches search for a relatively local optimum, so placing considerable weight on the assumed prior model structure and associated parameter values.

In the present case, on the advice of of David Leedal (pers. comm.), the former deterministic approach is used and all the parameter values, except ll, lo and CS, are specified by reference to the climate science literature. With the other parameter values fixed, ll, lo and CS are simultaneously optimized.
to minimize the sum of squares of the errors between the model output and
the measured GTA in Figure 1, resulting in a coefficient of determination
based on the simulated output of $R_T^2 = 0.835$, i.e. 83.5% of the measured
GTA variance is explained by the model. It must be emphasized, however,
that the parameter values obtained in this manner and shown in Table 1,
are supplied here simply for the information of the reader as an example
of how this particular version of MAGICC performs. However, the nature
of the parameter specification used here is not particularly important in
the present context because the model is only being used here to illustrate
the general similarities and differences between models obtained, as in this
case, by hypothetical deduction and those obtained by statistical induction
in subsequent sections of this report.

Figure 2: Response of the 80th order MAGICC climate model to a step input of total
radiative forcing compared with the step response of the associated 8th order DBM em-
ulation model, which is obscured because the responses are so similar, as shown by the
error plot in the top panel.

Figure 2 shows the step response of the MAGICC model with the pa-
rameter values presented in Table 1. Here, the step input of total radiative
forcing, shown as a dashed line, is changed from zero to 3.7 W/m$^2$. This is
the radiative forcing input change normally assumed to be consistent with
doubling the atmospheric $CO_2$ ($2 \times CO_2$) and so defining the climate sensi-
tivity, CS, as the equilibrium temperature change resulting from this change in atmospheric \( CO_2 \). The simulated global average temperature response in °C is shown as a full black line and we see that this response is both smooth and very slow, taking about 3000 years to reach the close vicinity of the steady state equilibrium level at 4.070, which defines CS in this case.

Despite the high order of the MAGICC model, with its 80th order nominally characterized by 80 eigenvalues that dictate its dynamic behaviour, the simulated step response is quite simple. In fact, if we are concerned with the input-output response of the model, then the model is only 43rd order because the transfer function between the radiative forcing and GTA exhibits ‘pole-zero’ cancellation, where 37 roots of the denominator polynomial are cancelled by identical roots of the numerator polynomial. In other words, the state-space form of MAGICC obscures a simpler underlying dynamic structure, the existence of which is revealed by the input-output transfer function form of the model. This transfer function form is not often alluded to by climate scientists, despite its importance in dynamic systems terms. It is important because, unlike the state space form, the transfer function is unique: namely, for an \( n \)th order model, there are an infinite number of state space realizations, depending on the definition of the state vector but all of these have the same \( n \)th order, input-output transfer function model, which is normally parametrically more efficient.

Also plotted as a black line in Figure 2, although almost completely obscured by the MAGICC model response, is the output of a reduced 8th order model that emulates the response of the 80th order MAGICC model almost perfectly: the error between the two model responses is very small and shown much enlarged (note the scale) in the upper panel. Figure 3 shows that this excellent emulation carries over the more discerning frequency response characteristics of the the two models, where the 8th order model almost completely matches the phase and gain properties of the MAGICC model. The left panel compares the standard gain-frequency plot of the MAGIC model with the same plot of the 8th order emulation model; while, in the right panel, the frequency axis is replaced by the time period (which can be interpreted as a local time constant) associated with the frequency, \( \tau_L = 2\pi/f \) years, where \( f \) is the frequency in radians/year.

The reduced 8th order model was obtained using a ‘balanced reduction’ technique for state-space models (see e.g. Varga, 1991; Gugercin and Antoulas, 2004), where the reduced order model is obtained analytically from the state-space model of the high order system. Consequently, this high order
model must be specified in a linear state-space form, as in the present case of the MAGICC model. The DBM model estimation procedures used in the next section can also be used for model reduction but the comparative results are not plotted in Figures 2 and 3 because they would also be obscured.

The DBM approach to large model emulation is quite different to balanced model reduction: in particular, it is based on statistically identifying the best identifiable low order model that explains the noise-free simulated output of the high order model and estimating the parameters in this identified model structure. Note that, in this case, only simulated data from the high order system is required for model reduction; its state space model does not have to be known. DBM model emulation was originally applied to large global carbon cycle simulation models (Young et al., 1996; Young, 1998) but has been further developed and applied to more general large simulation models.
since then (see e.g. Young and Ratto, 2009, 2011, and the prior references therein).

Figure 4: Reduced order emulation modelling. Left panel: variance of the error between the step responses of reduced order models and the full MAGICC model, for a range of reduced order model orders (red dots). Also shown for comparison are the DBM model emulation results (black dots). Right panel: reduced order model step responses (black lines) and the full MAGICC model step response (red line).

Figure 4 illustrates why the 8th order model was chosen for emulation. The error between the reduced order and MAGICC model responses is very small so the left hand panel is a logarithmic plot of the error variance against the model order. In the case of the DBM emulation (black dots) the model was not identifiable after 8th order but the error variance is by then is almost $10^{-15}$, quite a lot smaller than that achieved by balanced model reduction. The latter continues to reduce the error variance up to about 30th order but then this levels out, with no real improvement. The right hand panel of Figure 4 compares the step responses (black lines) for the reduced order models up to 43rd order: except for the first order model, these are all
visibly very similar to, and so mostly obscured by, the full MAGICC model step response (red line).

It is interesting to note that reduced order models can also be obtained from the responses of the AOGCM models using DBM emulation. For instance, in the case of the GFDL-R15-a version of the GFDL model, the emulation model identified from the simulated response to a radiative forcing step of $4 \times CO_2$, using the DBM identification and estimation methods outlined in the next section 5, is the following first order stochastic DBM model:

$$\frac{dx(t)}{dt} = -0.00133 x(t) + 0.626 \frac{du_1(t)}{dt} + 0.00158 u_1(t) + \eta(t) \quad (1)$$

where $x(t)$ is the GTA, $u_1(t)$ the total radiative forcing (in the simulation, this is a step input from zero to 3.656 Wm$^{-2}$) and $\eta(t)$ is a noise variable. This additive noise component is identified and modelled as a ninth order Auto-Regressive Moving Average (ARMA) model. A stochastic model is required in this case because the simulated output appears to have additive, coloured noise that cannot be accounted for by the effect of the deterministic input. Despite its low order, the deterministic, simulated output of this model explains 99.6% of the large model output variance. As we shall see in the next section, this model has a $2 \times CO_2$ climate sensitivity of $4.41 \pm 0.014$ and a time constant of 754 ± 31.4.

The GFDL-R15-a and other emulation modelling results shown in Figures 2 to 4 are interesting in their own right. However, their consideration in the present report is primarily because they show how the time and frequency responses of high order systems can be reproduced almost exactly by low order emulation models. This is the main reason why any attempt at estimating the parameters of the high order system encounters serious identification problems: there is just not enough information in the observed data to identify anything more than that revealed by the DBM method of model reduction. This arises because the dynamic response of the high order model is influenced primarily by a small number of ‘dominant modes’ (see e.g. Liaw, 1986; Young, 1999), the combined effect of which dominates the observed response of the model. As Jarvis and Li (2011) have pointed out, it is also the reason why Hasselmann et al. (1993, 1997) found that the simulated response to $2 \times CO_2$ radiative forcing of the ECHAM3 model was captured by the sum of just three first order exponential terms; whilst Hooss (2001) and Lowe (2003) found that, in the case of both the ECHAM3 and
HadCM3 models, the response to $4 \times CO_2$ forcing was captured by the sum of two first order exponential terms.

It is this dominant modal behaviour that most influences the observed response of the high order model and, if this model adequately characterizes the dynamics of the real climate system, then they will also be very important in their effect on the observed real system response, at least in the short to medium terms. For this reason, any study, such as that reported in the next section, that seeks to identify a stochastic model of globally averaged temperature variations for the purposes of short to medium term forecasting or emissions management, should seek an identifiable, low order model that captures the dominant modal behaviour and is also optimal in statistical terms.

3 Second Order Emulation of MAGICC

Before proceeding to look at real data, let us consider the second order DBM emulation of the 80th order MAGICC. This is the simplest reduced order model that is able to explain the MAGICC response well, as shown in Fig.5

Figure 5: Second order DBM emulation of the 80th order MAGICC model.

where the step response of the second order model is compared with the step
response of the MAGICC model. Here the coefficient of determination based on the simulated model output of $R_T^2 = 0.999$; i.e. 99.9% of the MAGICC model variance is explained by this estimated second order model, which takes the following form:

$$\frac{d^2 x(t)}{dt^2} + 0.0511 \frac{dx(t)}{dt} + 0.000126 \ x(t) = 0.153 \ \frac{d^2 u(t)}{dt^2} + 0.0159 \ \frac{du(t)}{dt} + 0.0000524 \ u(t)$$

This model is characterised by two real eigenvalues at $-0.048$ and $-0.0026$ that determine its dynamic response characteristics, with residence times of $1/0.0485 = 20.6$ and $1/0.0026 \approx 385$ years, respectively.

Using partial fraction expansion, the transfer function (see next section 4) associated with the model (2), can be decomposed conveniently, as shown in Fig.6, into a form where the radiative forcing $u(t)$ travels through three parallel pathways: one that has an instantaneous effect on the atmospheric temperature (i.e. within one year); a second which represents the ‘fast’ response with residence time 20.6 years; and a third, accounting for the ‘slow’ response of 385 years. The initial rapid response of the model, represented

Figure 6: Decomposition of DBM emulation of the 80th order MAGICC model.
by the instantaneous and fast pathways, accounts for 76% of the radiative forcing effect on the atmospheric temperature; while the slow pathway’s effect is only 24%. In other words, the ‘short term’ response observed over the recorded historical period from 1850 to the present is likely to dominated by the two fast pathways, with only small evidence of the long term behaviour. When we consider the analysis of the real data in the next section 4, will see that this is indeed the case.

4 DBM Model Identification and Estimation

The general consensus amongst climate scientists is that an appropriate model relating global climate variables, such as those shown in Figure 1, should be formulated in terms of differential equations. For example, the MAGICC model considered in the previous section involves interconnected set of 80 differential equations that form a set of continuous-time state equations and we have seen that the observed dynamic behaviour of this model can be emulated almost exactly by a much lower order DBM differential equation model.

A typical DBM model of this type is normally presented in a useful Transfer Function (TF) form but, to start with, these will be considered also in the equivalent differential equation form that is more familiar to a wider audience. In the case where a single output variable $x(t)$ is related to $p$ input variables $u_i(t), i = 1, 2, \ldots, nu$, the model takes the following general form (see e.g. Young, 2011, here and below):

$$\frac{d^n x(t)}{dt^n} + a_1 \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + a_n x(t) =$$

$$\sum_{i=1}^{nu} b_{0i} \frac{d^{m_i} u_i(t - \tau_i)}{dt^{m_i}} + \cdots + b_{mi} u(t - \tau_i)$$

(3)

where $\tau_i, i = 1, 2, \ldots, nu$ are pure time delays to allow for any such delays in the system dynamics; while $a_j, j = 1, 2, \ldots, n$ and $b_{ji}, j = 0, 2, \ldots, m; i = 1, 2, \ldots, nu$ are normally constant parameters. A model such as this can be transformed straightforwardly into an equivalent state-space form. More importantly in the present context, it can also be written in the equivalent TF form:

$$x(t) = \sum_{i=1}^{nu} \frac{b_{0i}s^{m_i} + b_{1i}s^{m_i-1} + \cdots + b_{mi}u_i(t - \tau_i)}{s^n + a_1s^{n-1} + \cdots + a_n} = \sum_{i=1}^{nu} \frac{B_i(s)}{A(s)}u_i(t - \tau_i)$$

(4)
where $s^r$ is the derivative operator, i.e. $s^r = d^r/dt^r$ and $\{B_i(s); A(s)\}, i = 1, 2, \ldots, nu$, are the appropriately defined polynomials in this operator. A multi-output system can be defined as an amalgamation of single output models such as this.

Equation (1) is a simple example of (4) with $p = 1$, $n = 1$, $m = 2$ and $\tau = 0$. Its TF equivalent is

$$x(t) = \frac{0.626s + 0.00158}{s + 0.00133}u_1(t)$$

(5)

where it will be noted that it is easy to calculate the dynamic characteristics of the model: the steady state gain is obtained when $s = d/dt = 0$, i.e. $G_\infty = 0.00158/0.00133 = 1.19$ and the time constant is $T = 1/0.00133 \approx 750$ years. It is widely accepted that the radiative forcing for a doubling of CO2 is likely to be about $5.35 \log_e(2) = 3.7 \text{ Wm}^{-2}$ (Myhre et al., 1998) so that, in this case, the nominal $2 \times \text{CO}_2$ climate sensitivity for this model is $3.7G = 4.4$.

Equation (2) can also be considered in the following TF form, this time with with $p = 1$, $n = 2$, $m = 3$ and $\tau = 0$:

$$x(t) = \frac{0.153s^2 + 0.0159s + 0.0000524}{s^2 + 0.0511s + 0.000126}u_1(t)$$

$$= \frac{0.153s^2 + 0.0159s + 0.0000524}{(s + 0.048)(s + 0.0026)}u_1(t)$$

(6)

Then, using partial fraction expansion (the residue routine in Matlab), the resulting parallel pathway model can be written as:

$$x(t) = 0.153u_1(t) + \frac{0.00782}{s + 0.0485}u_1(t) + \frac{0.000265}{s + 0.0026}u_1(t)$$

$$= 0.153u_1(t) + \frac{0.0161}{1 + 0.001}u_1(t) + \frac{0.102}{1 + 384.7}u_1(t)$$

(7)

which is the model on which the mechanism shown diagrammatically in Fig.6 is based.

The TF form (4) is convenient for introducing stochastic inputs. The most common way of doing this in the case of sampled data is to add a noise or uncertainty component $\xi(k)$ to the output, where the argument indicates the value of $\xi$ at the $k$th sampling instant. This additive noise is normally assumed to be coloured noise, modelled as a discrete-time AutoRegressive
Moving Average (ARMA) process (see Box and Jenkins, 1970; Young, 2011), so that the complete stochastic model can be written informally\(^1\) as follows:

\[ x(t) = \sum_{i=1}^{n} \frac{B_i(s)}{A(s)} u_i(t - \tau_i) \]

\[ \xi(k) = \frac{D(z^{-1})}{C(z^{-1})} e(k); \quad e(k) = \mathcal{N}(0, \sigma^2) \] (8)

Observation Equation: \( y(k) = x(k) + \xi(k) \)

Here, \( e(k) \) is a zero mean, serially uncorrelated sequence of random variables (white noise), while \( C(z^{-1}) \) and \( D(z^{-1}) \) are the following polynomials:

\[ C(z^{-1}) = 1 + c_1 z^{-1} + c_2 z^{-2} + \ldots + c_p z^{-p} \]
\[ D(z^{-1}) = 1 + d_1 z^{-1} + d_2 z^{-2} + \ldots + d_q z^{-q} \] (9)

where \( z^{-r} \) is the backward shift operator, i.e. \( z^{-r} \xi(k) = \xi(k - r) \).

The total model structure is specified by the vector

\[ [n \ m \ (i = 1, 2, \ldots, nu) \ \tau_i \ (i = 1, 2, \ldots, nu) \ p \ q] \]

Identification of the model structural parameters in this vector, as well as the estimation of the polynomial parameters \( (a_i, \ i = 1, 2, \ldots, n, b_{i,j}, \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, nu, c_i, \ i = 1, 2, \ldots, p, d_i, \ i = 1, 2, \ldots, q \) and \( \sigma^2 \) that characterize the associated model, is a well known statistical problem. In DBM modelling, it is normally solved by maximum likelihood estimation using the Refined Instrumental Variable method (see e.g. Young, 2011, 2015) for ‘hybrid’ continuous-time models of this type (RIVC). This is available as the `rivcbjid` and `rivcbj` routines in the CAPTAIN Toolbox\(^2\) for Matlab\(^\text{TM}\), which is used in all the DBM modelling considered in subsequent sections.

One advantage of the continuous-time TF model is that it can be related directly with the Laplace transform transfer function form of differential equation models. Indeed, \( s \) is often used as the Laplace operator and it is used here as the derivative operator to emphasize this relationship. If there are no initial conditions affecting the data, then the Laplace transform and derivative operator forms of the TF are identical. If there are initial

\(^1\)Informal because it involves a mixture of the two operators \( s \) and \( z^{-1} \).

\(^2\)The CAPTAIN Toolbox is freely available via the website [http://captaintoolbox.co.uk/Captain_Toolbox.html/Captain_Toolbox.html](http://captaintoolbox.co.uk/Captain_Toolbox.html/Captain_Toolbox.html)
conditions effects present, however, the Laplace transfer of the differential
equation introduces additional terms that are dependent on the initial con-
ditions of the output, input and their time derivatives. This is discussed in
Appendix A, which shows how accounting for the initial conditions is not
difficult and can be used to good effect in the model estimation (as we shall
see later in section 5.2).

In the subsequent DBM modelling, the statistically nonstationary inputs
to these DBM models (total radiative forcing, \( CO_2 \) forcing and the Atlantic
Multidecadal Oscillation (AMO) index, as plotted in Figure 1) are modelled as
discrete-time univariate time series using the Dynamic Harmonic Regression
(DHR) algorithm, which is also available in the CAPTAIN Toolbox as the
routine dhr. The basic DHR model is an Unobserved Component (UC) model
that contains non stationary (time variable parameter) trend, cyclical, and
irregular components:

\[
y(k) = T(k) + C(k) + e(k) \tag{10}
\]

Here, the trend is modelled by one of several random walk models, such as
integrated or smoothed random walks, and these serve to estimate long term
changes in the level of the series. The cyclical term \( C(k) \) is modelled as

\[
C(k) = \sum_{i=1}^{R_s} \{a(i,k) \cos(\omega_i k) + b(i,k) \sin(\omega_i k)\} \tag{11}
\]

where each \( a(i,k) \) and \( b(i,k) \) is a stochastic Time Variable Parameter (TVP),
again modelled by one of the random walk family, and \( \omega_i, i = 1, 2, ..., R_s \),
are the fundamental and harmonic frequencies associated with the periodicity in
the series. The frequency values are chosen by reference to the spectral prop-
ties of the time series, as discussed later. The trend component can also be
considered as a stochastic, time variable intercept parameter in the DHR and
so it could be incorporated, if so desired, into the cyclical component as a
zero frequency term (i.e. by introducing an extra term in the summations at
\( i = 0 \) with \( w_0 \) or \( f_0 \) set equal to zero). This DHR model can be considered as
a straightforward extension of the classical, constant parameter, Harmonic
Regression (or Fourier series) model, in which the gain and phase of the har-
monic components can vary as a result of estimated temporal changes in the
parameters \( a(i,k) \) and \( b(i,k) \).

The DHR model is estimated using the Kalman filter (KF) and associ-
ated recursive Fixed Interval Smoothing (FIS) algorithms, with the ‘hyper-
parameters’ (noise-variance ratios associated with the random walk models)
optimized to minimise the difference between the spectrum of the DHR model and the empirical spectrum of the series being modelled: the details are given in Young et al. (1999) and, from a wider DBM perspective, in Young (2011).

4.1 Hypothetico-Inductive DBM Modelling

Obviously, hypothetico-deductive and inductive methods of modeling are not mutually exclusive and HI-DBM modelling (Young, 2013) is an attempt to meld together the best aspects of both approaches and produce a systematic approach to model development. It is a logical extension of DBM modeling when elements of the purely inductive DBM model are not considered fully satisfactory for some reason; and also a way of investigating existing conceptual models to see if they can be improved using DBM models and tools. In any specific application, HI-DBM modeling is applied to an existing set of input-output data and it normally consists of three stages:

1. One or more selected hypothetico-deductive conceptual models that have been proposed previously are investigated in detail, where necessary and if it is possible, re-optimizing the model parameters in order to better understand the conceptual model characteristics and/or to remove any ambiguities in the previously published literature on the model.

2. The same data are then analyzed using standard DBM modeling procedures and the validated DBM model is compared with the conceptual model in order to investigate the similarities and differences between the two models.

3. Based on the results obtained in stage 2, it is decided whether any elements of the existing conceptual model can be incorporated within the standard DBM model framework in order to improve its physical interpretability and/or whether the conceptual model can be modified in order to improve its structure and explanation of the data.

In the present context of climate modelling, we have effectively considered stages 1 in section 2. This supports the low order models that derive from the inductive DBM modelling process and, at the conceptual level, it suggests that the radiative forcing measures produced by the climate science community should be considered as inputs to DBM models. The second and third stages are considered in the next section 5.
5 HI-DBM modelling of Global Average Temperature

The data plotted in Figure 1 are typical large scale climate series that relate to the variations in the Global Temperature Anomaly (GTA). The first panel is a plot of the GTA in °C, from 1850 to 2011, based on the deviations about a reference level defined by the average temperatures between 1951 and 1980; these were downloaded from the Carbon Dioxide Information Analysis Center (CDIAC) website at http://cdiac.ornl.gov/trends/temp/jonescru/jones.html. The second panel is a plot of the total radiative forcing signal; and the third panel is the CO₂ component of this total signal. These were downloaded from the Potsdam Institute for Climate Impact Research website at http://www.pik-potsdam.de/~mmalte/rcps/. Finally, the lower panel is a plot of the Atlantic Multidecadal Oscillation (AMO) index. Although nominally this index relates to the climate variations in the Atlantic region, it is used in section 5 to help explain aspects of the GTA. This series was downloaded from the Physical Sciences Division, Earth System Research Laboratory of NOAA at http://www.cgd.ucar.edu/cas/catalog/climind/AMO.html.

5.1 Initial DBM Modelling and Forecasting

The first step in the DBM approach to inductive modelling of data such as those shown in Figure 1 is the definition of the model and the statistical identification of the model dynamic order. The statistical model order identification procedure is fairly standard: a range of possible model orders are investigated and evaluated in relation to well known model order identification criteria using the CAPTAIN routine rivcbjid. The identification criteria include the Akaike Information Criterion (AIC) suggested in an early seminal paper by Akaike (1974) and a Bayesian extension of the AIC criterion: the Bayesian Information Criterion (BIC). AIC and BIC both work well in practice, although the AIC is more prone to over-fitting. Section 5.4.5 of the excellent book by Priestley (1981) provides a very good review of these and other identification procedures. The YIC (Young, 1989) is a heuristically defined identification criterion of a different type that exploits the special properties of the Instrumental Product Matrix (IPM), so it can only be used with IV-type algorithms such as RIVC.
Considering first the identification of a single input, single output model between the total radiative forcing, denoted by \( u_1(t) \), and the GTA, denoted by \( x(t) \), both the YIC and AIC criteria are minimized for the first order \([1 1 2 0 0 1 0]\) model, which is estimated as follows:

\[
x(t) = \frac{0.036(0.010)}{s + 0.060(0.023)} u_1(t) \quad (i)
\]

\[
\xi(k) = \frac{1}{1 - 0.537(0.067)z^{-1}} e(k); \quad e(k) = \mathcal{N}(0, 0.008) \quad (ii)
\]

\[
y(k) = x(k) + \xi(k) \quad (iii)
\]

where the figures in parentheses adjacent to the parameter values are the estimated standard errors. In fact a second input \( U_{ic} = 1 \forall t \) is introduced in order to allow for the estimation of initial condition effects (see Appendix A and later section 5.2) but the details of the TF for this second input are not reported here to avoid any confusion. The steady-state gain, climate sensitivity and time constant associated with this model are 0.603, 2.23°C and 16.6 years, respectively.

Standard auto and cross correlation tests on the residual series \( e(k) \) in (12) yield acceptable results: it is not either significantly autocorrelated or correlated with the radiative forcing input series; and Jarque-Bera analysis (Jarque and Bera, 1987) confirms that its histogram has an acceptably normal distribution. The model explains the GTA reasonably well, with a coefficient of determination, based on the error \( \xi(k) = y(k) - \hat{x}(k) \), where \( \hat{x}(k) \) is the sampled, deterministic output of the model (12)(i), of \( R_T^2 = 0.842 \), which is a little better that the \( R_T^2 = 0.835 \) for the conditionally optimized, 80th order MAGICC model and its 8th order DBM emulator (see section 2). The lower right panel in Figure 7 compares the deterministic output of the model with the measured GTA, while the other panels show the contributions to this full model output from the atmospheric \( CO_2 \); the other radiative forcing contributions with the \( CO_2 \) forcing omitted; and the initial condition effect.

At first sight, the initially estimated TF model seems well identified and acceptable. However, despite the apparently reasonable diagnostic tests on the model residuals, the unexplained error \( \hat{\xi}(k) \) is unacceptably large and deserving of further investigation. First, it is interesting to examine whether there is statistical evidence of nonstationarity in the form of parameter variation. One approach is to apply Dynamic Transfer Function (DTF) estimation, utilizing the \texttt{dtfmopt} and \texttt{dtfm} routines in CAPTAIN applied to the
Figure 7: Contributions to the deterministic model output: $CO_2$ forcing, top left; other forcing, top right; initial condition effect, lower left; full deterministic model output compared with measured data, lower right.

discrete-time version of the model (12). The former indicates that there is no evidence for any variation in the time constant of the model but that there is some suggestion of variation in the steady state gain and, therefore, the $2 \times CO_2$ climate sensitivity. Figure 8 shows the resulting recursive estimate of CS that produces an $R_T^2 = 0.924$, which is similar to constant parameter models considered later in section 5.3.

The estimated CS in Figure 8 is changing considerably about the value estimated for the constant parameter model (12) (dash-dot line), with quite uncertain variations from around zero to the values around $4^\circ C$ of the climate models considered in section 2. It is also interesting to note that the large values are in the period between 1910 and 1940, with a big reduction around 1945. However, the associated constant time constant is estimated as only two years, which is very low from a climate dynamics perspective, so these results and observations should be interpreted with caution. Basically, they suggest that the model (12) is too simplistic and the data need to be explained
Looking again at the error series $\hat{\xi}(k)$, the discerning reader might notice a long-term pattern in the series. This is confirmed by noting that there is a significant reduction in the AIC at AR(29), with the AR(29) spectrum revealing a fairly sharp peak at a period of 51.6 years. If, therefore, the DHR algorithm is applied to the error series, with its hyper-parameters optimized so that its pseudo-spectrum matches the AR(29) spectrum, it is able to extract this quasi-periodic component, which explains a significant part of the error series, as shown by the full black line Figure 9. Here, the dash-dot line is the one year-ahead predictions from the Kalman filter, based on the full stochastic DHR model.

If this estimated quasi-cyclic component $\hat{\xi}_q(k)$ is now added to the TF model to further explain the error residuals, then the final model between the total forcing and the GTA takes the following form, where it will be noted that the parameter estimates for the system part of the model are not
Figure 9: DHR model based on a major 51.6 year period component (full black line) and other, much lower amplitude, components, at periods of 20.0, 9.26, 3.61 years. Forecast shown from 2011 to 2050. The dash-dot line shows the one-year-ahead predictions using the full stochastic model).

changed very much but their standard errors are significantly reduced:

\[ x(t) = \frac{0.0358 \pm 0.002}{s + 0.059 \pm 0.004} u_1(t) \]  
(1)  

\[ \xi(k) = \hat{\xi}_q(k) + \frac{1}{1 - 0.167(\pm 0.078)z^{-1}} e(k); \quad e(k) = \mathcal{N}(0, 0.001) \]  
(ii)  

\[ y(k) = x(k) + \xi(k) \]  
(iii)  

The associated graphical output from this model is shown in Figure 10. The left panel compares the deterministic and stochastic outputs from the model with the measured GTA; while the right panel compares the deterministic output with the GTA modified to remove the estimated quasi-cyclic component. Here, the coefficient of determination based on the error between the deterministic output and this modified series is \( R^2_T = 0.978 \) and the standard coefficient of determination based on the one-year-ahead prediction errors is \( R^2 = 0.979 \). In other words, the deterministic part of the model, including the estimated quasi-cyclic component, explains much of the variance in the GTA series.
The main TF model component \((i)\) in (13) can be written in the alternative form
\[
x(t) = \frac{G}{1 + Ts} u_1(t)
\]
where \(G = 0.0358/0.059 = 0.607\) is the steady state gain and \(T = 1/0.059 = 16.95\) years is the time constant. Using Monte Carlo analysis, based on the estimated error covariance matrix of the directly estimated parameters and 10000 realizations, the uncertainty in these estimates is approximately normally distributed, with \(\hat{G} = 0.607(0.003)\), \(\hat{T} = 16.95(0.25)\) years and the associated \(2 \times CO_2\) climate sensitivity \(\hat{CS} = 2.243(0.011)\)°C, where the figures in parentheses are the estimated standard errors. \(\hat{CS}\) is considerably different from the climate model sensitivities encountered in section 2 but it is within the usually accepted range of 1.5 to 4.5 °C (see e.g. Cubasch and Coauthors, 2001). It also agrees reasonably with the values obtained using econometric
time series analysis (see e.g. Kaufmann et al., 2006a; Mills, 2009, where
the former cite a range from 1.7 to 3.5°C; and the latter gives an estimate
of 2.16 ± 0.44 °C). The estimated time constant is much less than that of
the TF model (5) for the GFDL data set (500 years); and the eight time
constants associated with the DBM emulation of the MAGIC model have
time constants ranging from 0.28 to 647 years, where the closest in value,
at 17.7 years, does not have the most dominant effect. The main problem
in this case is, of course, the limited information on such long time constant
behaviour contained in the 162 years of data used in the analysis (Figure 1).
Also, the radiative forcing may not be ‘sufficiently exciting’ (see e.g. Young,
2011) to allow for the identification of any longer term modes (see also the
discussion in section 2 of Raupach, 2013).

Bearing these identifiability issues in mind, it is best to assume that the
parameters only reflect the short term dynamics and should be considered
with this in mind; for example, the associated climate sensitivity estimates
are likely to be ‘short term’ measures and may not reflect the ‘long-term’ sen-
sitivity. Note, however, that any conventional hypothetico-deductive mod-
elling will suffer from similar identifiability problems, with the long term
behavior of the models dominated by the modeller’s hypotheses concerning
the nature of the climate system and its long term dynamics.

As a result of these considerations, the full stochastic DBM model (13) is
intended mainly for short term forecasting and it produces quite good results
in this regard, as shown in Figures 32 and 12. The first of these is a forecast
from 2000 to 2030 based on the model estimated in 2000 from the historical
data up to 2000. Obviously, to be a true ex-ante forecast, this requires a ten-
year-ahead prediction of the changes in the total radiative forcing \( u_1(t) \) over
this period. This is obtained, once again, using the DHR algorithm, this time
based on an AIC identified AR(17) spectrum, but now used to model the non
stationary behaviour of the sampled \( u_1(k) \) series. This is then interpolated
automatically by the continuous-time simulation routine lsim in Matlab, to
form \( u_1(t) \). Up to the 30-year-ahead forecasting origin, the one-year-ahead forecasts are compared with the measured GTA, with the 95% confidence
bounds on the forecast shown in grey. For the first ten years after the fore-
casting origin in 2000, the forecast can be compared with the measured GTA
and we see that the forecast has successfully captured the ‘levelling out’ of
the GTA that has occurred since the start of the new Millenium. After this,
the forecast GTA is starting to rise again, although with a shallower trend.
However, the confidence bounds are rapidly growing wider, reflecting the un-

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certainty in both the model itself and in the uncertainty associated with the forecast of $u_1(t)$.

Figure 11: Forecasting results obtained using the model (13): one-year-ahead forecasts up to 2000 and 30-year-ahead thereafter.

Figure 12 illustrates how the model (13) can be used to generate a sequence of ‘rolling’ forecasts, with the forecast origin being advanced each time by ten years; the DBM model parameters re-estimated up to this origin; and the required $u_1(t)$ input forecast generated by an updated DHR model. The fifth panel from the top is same forecast as that shown in Figure 32 but with only the first ten-years of the forecast plotted. However, the lowest panel presents the 200-year-ahead forecast from 2011 to 2210, based on all of the data available up to 2011. While it is unrealistic to use the model for such a long-term forecast, it does serve to illustrate the very uncertain nature of this forecast (and indeed, any forecast) over such a long time ahead, with predictions ranging from temperatures less that any encountered since 1850 to significantly greater than 3°C. Not surprisingly, this plot also shows how the
model expects the quasi-cyclic behaviour to continue. But, given the uncer-
tainty in this periodic component, the forecast contains only the underlying periodic signal defined by the peak in the AR(29) spectrum, as estimated up to the start of the forecasting period. In this connection, note that the DHR model for $u_1(k)$ was estimated here by deliberately excluding any damping of the periodic components, in order to illustrate better its underlying forecast form. If damping had been allowed then the amplitude of this cycle would have attenuated over the forecasting period.

Finally, while the model (13) produces good *ex ante* forecasting results, we need to consider whether it is credible in mechanistic modelling terms and satisfies the DBM modelling requirements in this regard. It is a linear differential equation model of the same basic form as traditional climate models and, although it is only first order, this is consistent with transfer function (1) estimated from the very large AOGCM climate model data. Also the parameters of the model are reasonable in physical terms as we see from its estimate of 2.243 for the $2 \times CO_2$ climate sensitivity. And while its time constant is considerably different from the dominant time constants of the high order climate simulation models, the parameters of these high order models are not estimated directly from the 162 years of globally averaged data set, with its limited statistical information on large time constants (see next section 5.2), so we might expect some differences in this regard. Nevertheless, the fact that the estimated DBM model does not contain any long term mode needs further evaluation and this is considered in the next sub-section 5.2.

The main limitation of the model in DBM terms is clearly the quasi-cyclic DHR component, which is a ‘black-box’ model and so difficult to justify in mechanistic terms. Moreover, its inclusion raises the coefficient of determination from 0.842 to 0.978, so it plays quite a large part in explaining the GTA data. It would help to justify the model further, therefore, if an additional input could be found that helps to explain the quasi-cyclic behaviour and so replaces the need for this DHR component. Although not essential from a forecasting standpoint, this is considered in section 5.3. Before this, however, it is worth considering whether there is any evidence at all in the data on long-term behaviour that might help to relate the DBM model with the MAGICC climate model considered previously in sections 2.1 and 3.

### 5.2 HI-DBM Modelling: Long Term Behaviour?

As outlined in sub-section 4.1, HI-DBM modelling involves a comparison of the purely inductive DBM model, in the present example the model (13),
with any available conceptual models. The most obvious model to consider in this regard is the reduced order MAGICC model (7). The obvious difference between these two models is their dynamic order: the latter is second order with both fast and slow modes of dynamic behaviour; while the former is first order, with a single fast mode.

In identifying the model (13), the question of initial conditions was not discussed because it was a standard approach in which the denominator of the TF associated with the initial conditions is the same as that on the main TF model (see Appendix A). Here, we will take a less conventional approach and allow the initial condition transfer function to have a different denominator. With this change, the resultant estimated model now takes the following form:

System Model
\[ x(t) = \frac{0.070s + 0.032}{s + 0.044}u(t) - \frac{0.40s + 0.006}{s + 0.009}u_{IC}(t) \]

AR Noise Model
\[ \xi(k) = \hat{\xi}_q(k) + \frac{1}{1 - 0.507z^{-1}}e(k); \quad e(k) = N(0, 0.0075) \] (15)

Output Equation
\[ y(k) = x(k) + \xi(k) \]

where \( u_{IC}(t) = 1.0 \forall t \) and, once again \( \hat{\xi}_q(k) \) is the quasi-cyclic component estimated by DHR analysis. In order to see whether this model relates to (7), the first term on the right hand side of the first equation can be decomposed and the resultant equation, manipulated to reveal the residence times, takes the form:

\[ x(t) = \{0.070u_1(t) - 0.400u_{IC}(t)\} + \frac{0.66}{1 + 22.8s}u_1(t) - \frac{0.282}{1 + 106.2s}u_{IC}(t) \] (16)

Direct comparison of this equation with (7) is not possible because of the unit \( u_{IC}(t) \) input but we see that this term has revealed some long-term behaviour in the data. Indeed, this has improved the explanation of the temperature variations a little, with \( R_T^2 = 0.854 \) compared with \( R_T^2 = 0.842 \) for the model (13) and \( R_T^2 = 0.80 \) for the MAGICC and reduced order MAGICC models. The deterministic output response on these models are compared in Fig.13, which also reports the poorer results obtained when estimating a discrete-time rather than a continuous-time model model.

It is important to note that the \( u_{IC}(t) \) TF in the model (15) is inserted primarily to account for initial condition effects and its long term dynamic
Figure 13: MAGICC, reduced MAGICC and DBM model comparisons.

Figure 14: Uncertainty in derived parameters obtained by MCS analysis.

effect is ill-defined, as shown in the Monte-Carlo Simulation (MCS) results of the Fig.14, where there is a very long tail on this slow residence time
distribution reaching in excess of 1000 years. The figure also shows the distributions of the fast residence time and the climate sensitivity for this model.

Considering the above results and those in the previous section 5, we see that the nominally ‘black-box’ DBM model satisfies DBM modelling requirements by explaining the data rather better than the climate models and being credible in physical terms. First, it is a continuous-time model similar to the differential equation for a ‘mixing process’ or ‘mixing box’ of the kind used in climate modelling. Second, the climate sensitivity of $CS = 2.8^\circ C$ (with bounds from 1.78 to 4.25: see Fig.14) for this model compares with MAGICC’s $4.4^\circ C$ and, as well as being within the usually accepted range of 1.5 to $4.5^\circ C$ (Cubasch et al, 2001), it is close to that of $2.14^\circ C$ obtained by Mills (2009). Third, the instantaneous response and residence time of the DBM model mixing process: $0.07^\circ C$ (bounds: 0.062 to 0.078) and 22.8 years (bounds: 17.3 to 33.6) compare with MAGICC fast mode values ($0.068^\circ C$ and 20.6 years), which account for 76% of the MAGICC response. Finally, the long period initial condition response suggests that there is some evidence of a poorly defined, long term mode with slow residence time $> 100$ years but that this is difficult to identify and estimate as a mode in the input-output model.

5.3 HI-DBM Modelling: A Quasi-Cyclic Input?

In one sense, The DBM models described by equations (13) and (15) are also HI-DBM models because the total radiative forcing input $u_1(k)$ used in their identification and estimation is supplied by climate scientists and based upon reasonable scientific assumptions about such forcing. But can we improve the HI-DBM nature of the model still further by finding a climate related input variable that exhibits quasi-cyclic behaviour with a period around 52 years and so negate the need for the DHR component in the DBM models (13) and (15)? One such variable is clearly the AMO index series shown in the bottom panel of Figure 1, which is quite highly correlated with the extracted quasi-cyclic component, as shown in the cross-correlation plot of Figure 15, where the AMO index is denoted by $u_2(k)$. Another measure of strong correlation between the signals is the magnitude squared coherence, defined as the cross power spectral density between the $u_2(k)$ and $\hat{\xi}(k)$, divided by the product of their individual power spectral densities. This has a noticeable peak of 0.64 at a period of 59.8 years. Given these high correlations, an obvious approach
is to evaluate whether it is possible to identify and estimate a differential equation model between $u_2(k)$ and $\hat{\xi}(k)$, which can then replace the DHR model for $\hat{\xi}(k)$ in (13). This is indeed possible and the resultant second order model between $u_2(k)$ and $\hat{\xi}(k)$ is identified with the structure $[3 \ 4 \ 4 \ 0 \ 0 \ 17 \ 0]$ where, once again, a second unitary input is introduced to estimate and account for the initial condition effects. The model, without this second input, takes the form:

$$x_2(t) = \frac{b_{20}s^2 + b_{21}s + b_{22}}{s^3 + a_{21}s^2 + a_{22}}u_2(t) + \eta(t)$$

$$\eta(k) = \frac{1}{1 + c_1z^{-1} + c_2z^{-2} + \cdots + c_{17}z^{-17}}e(k); \quad e(k) \sim \mathcal{N}(0, \sigma^2);$$

(17)

Here, $x_2(t)$ denotes the new system variable that is generated from the input $u_2(t)$ and replaces $\hat{\xi}(k)$ in the model. The explanation of $\hat{\xi}(k)$ is reasonable, as shown in Figure 16, where the coefficient of determination based on simulated deterministic output of the model is $R^2_T = 0.57$; and the standard coefficient of determination, based on the one year ahead prediction errors, is $R^2 = 0.80$. 

Figure 15: Cross-correlation function between the Atlantic Multidecadal Oscillation (AMO) index, $u_2(t)$, and the estimated cyclic component $\xi(t)$ of GTA.
Figure 16: Estimate of the quasi-cyclic component of the Global Temperature Anomaly (GTA), based on the differential equation model between the Atlantic Multidecadal Oscillation (AMO) index and the component. Also shown are the one year ahead predictions using the full model with an AR(17) noise model.

The final HI-DBM model simply combines the two components from (13) (i) and (17) to form:

\[
x_1(t) = \frac{b_{10}}{s + a_{11}} u_1(t) \quad (i)
\]

\[
x_2(t) = \frac{b_{20}s^2 + b_{21}s + b_{22}}{s^2 + a_{21}s + a_{22}} u_2(t) \quad (ii)
\]

\[
y(k) = x_1(k) + x_2(k) + \frac{1}{1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_{17} z^{-17}} e(k) \quad (iii)
\]

\[
e(k) = \mathcal{N}(0, \sigma^2); \quad (18)
\]

The parameters in this model are optimized using a ‘back-fitting’ method of maximum likelihood estimation (see e.g. Young, 2011, section 7.6, Chapter 7) and Figure 17 shows the simulated deterministic contributions \(x_1(t)\) and \(x_2(t)\), as well as the total output \(x(t) = x_1(t) + x_2(t)\) generated by this
Figure 17: The left and centre panels show the deterministic contributions of the two inputs $u_1(t)$ and $u_2(t)$ to the output temperature, shown in the third panel and compared with the measured data. The 95% uncertainty bounds quantify the effects of all estimated parametric uncertainty and the three stochastic inputs.

An optimized model in the right hand panel. Figure 18 is an enlargement of the right hand panel of Figure 17. We see that $x(t)$ satisfactorily blends the dynamic effects of the two inputs $u_1(k)$ and $u_2(k)$ to provide a reasonable explanation of the GTA, with an $R^2_T = 0.923$. This is not such a good explanation of the GTA as the forecasting model considered in the previous section ($R^2_T = 0.979$) but it has the advantage of using the $u_2(t)$ input to explain the quasi-cyclic effects via the constant parameter model (18)(ii), so removing the need for the time variable parameter DHR model. It is, therefore, more acceptable from a DBM modelling standpoint.

The estimated model parameters of the model (18) are as follows:

\[
\hat{a}_{11} = 0.0627(0.0066); \quad \hat{b}_{10} = 0.0363(0.0029); \quad \hat{a}_{21} = 0.0167(0.0095);
\]
\[
\hat{a}_{22} = 0.0091(0.0008); \quad \hat{b}_{20} = 0.4409(0.0405); \quad \hat{b}_{21} = -0.0103(0.0028); \quad \hat{b}_{22} = 0.00226(0.0004); \quad \sigma^2 = 0.0034;
\]

The associated steady state gain, $2 \times CO_2$ climate sensitivity and time constant are 0.580, 2.14 °C, and 16 years, respectively. The uncertainty in these derived parameter estimates is evaluated using Monte Carlo analysis, based of the estimated parametric covariance matrix and 10,000 stochastic realiza-
Figure 18: The deterministic output of the full HI-DBM model compared with the global temperature anomaly measurements. The 95% uncertainty bounds quantify the effects of all estimated parametric uncertainty and the three stochastic inputs.

... this yields 95% confidence intervals on climate sensitivity of 2.09 to 2.21 °C; while on the time constant they are 14.7 to 17.5 years. These derived parameters estimates are similar the initial DBM model (13) and their uncertainty is shown graphically by the resulting normalized histograms of the latter two parameters plotted in Figure 19.

The model (18) is quite well defined statistically, as we see from various diagnostics:

1. First, the recursive estimates\(^3\) of the main model parameters \(a_{11}, b_{10},\) CS and \(\tau\) are plotted in Figure 20. Recursive estimation, which provides estimates of the the parameters sequentially through the data (under the standard assumption that they are constant) is an inherent advantage of RIVC model estimation. In this case, it shows that the parameter estimates have settled to reasonably stable levels by around 1970. There is an interesting upward movement in the derived recursive estimates of CS and \(\tau\) between 1940-1950, the significance of which is

\(^3\)Recursive estimation is an inherent, advantageous feature of the RIVC algorithm (see Young, 2011).
Figure 19: Monte Carlo evaluation of the HI-DBM model: Normalized histograms of the estimated $2 \times CO_2$ climate sensitivity and main time constant.

Figure 20: Recursive estimates of main HI-DBM model parameters.
unclear, although it does cover the period of the second World War.

2. Second, the final model residuals \( \hat{e}(k) \), plotted in the top panel of Figure 18, are reasonably white and conform to the standard correlation requirements: both the autocorrelation function of \( \hat{e}(k) \) and the cross-correlation between \( \hat{e}(k) \) and \( u_1(k) \) show no significant correlation at any lags. And while the cross-correlation between \( \hat{e}(k) \) and \( u_2(k) \) at lag one are 0.17 which marginally transgresses the standard error boundary (0.16) this is not exceeded at any other lags and all correlations are less than two standard deviations.

3. Based on the results from the \texttt{histon} routine in the CAPTAIN Toolbox, the histogram of \( \hat{e}(k) \) suggests a normal distribution and the Jarque-Bera statistic confirms this.

4. Finally, if the model is estimated on the data up to 1990, the estimated parameters of the main model are: \( \hat{a}_{11} = 0.0651(0.0090) \) and \( \hat{b}_{10} = 0.0391(0.0044) \), which yield \( \hat{CS} = 2.22^\circ \text{C} \) and \( \hat{\tau} = 15.4 \) years. This model explains the GTA up to 2011 almost as well as the model estimated over the whole data set, with \( R_T^2 = 0.918 \), providing a reasonable validation of the model.

5.4 Comments on the HI-DBM model

Although the model (18) is not intended for forecasting, it can be used in this role, as shown in Figure 21, which can be compared with Figure 32. Here, a forecast of the AMO input is required to produce the GTA forecasts and it is generated by a DHR model based on an AIC identified AR(23) spectrum. Overall, the HI-DBM model does not perform as well as the DBM model with an \( R_T^2 = 0.965 \), compared with \( R_T^2 = 0.98 \) for the DBM model, mainly because it does not yield quite such good one-year-ahead forecasts. However, between 2000 and 2011, the forecasts are actually better, with an error standard deviation of 0.064 compared with 0.118 for the DBM model. It’s main disadvantage is its relative complexity and robustness, when compared with the simpler DBM forecasting engine: this does not rely on the second AMO input and is also inherently adaptive because it exploits the DHR algorithm to forecast the quasi-cyclic component of the GTA.

One negative feature of the HI-DBM results is that the recursive estimates of the parameters in the \( x_2 \) model (18)(ii) are rather volatile and slow
to converge, as shown in Figure 22, suggesting possible time variable parameters or nonlinearity. This can be investigated using the dtfm time variable parameter estimation routine in CAPTAIN, although this is intended for discrete-time models that do not explain the data so well. However, the results suggest that the parameters may well be time variable, with the relatively smoothly changing parameter model explaining the data much better than the constant parameter model. An initial evaluation of the model in alternative State-Dependent Parameter (SDP) terms (see e.g. Young, 2000; Young et al., 2001), using the sdp routine in CAPTAIN, suggests that an SDP nonlinear model may also be worth considering further.

Despite its limitations, the constant parameter HI-DBM model (18) still demonstrates well that the quasi-cyclic behaviour in the GTA can be explained statistically by the AMO. Consequently, it should be considered mainly as a vehicle for further research on the nature of the dynamic relationship between the globally averaged climate variables. This could include the consideration of other climate variables that may help to explain the quasi-cyclic behaviour or may be influencing both AMO and the quasi-cyclic component of the GTA. It could also include studies to investigate whether the present black-box model (18)(ii) between $u_2(t)$ and $x_2(t)$ can be
interpreted in a physically meaningful manner.

In this latter regard, it is interesting to note that the second order model (18)(ii) between $u_2(t)$ and $x_2(t)$ can be decomposed into a feedback connection of two first order processes, as shown in Figure 23. This decomposition

\[ u_2(t) \rightarrow + \rightarrow \frac{bs + c}{s + a} \rightarrow x_2(t) \]

\[ \frac{d}{s + f} \]

Figure 23: Feedback interpretation of the model between $u_2(t)$ and $x_2(t)$ obtained by block diagram decomposition.
is ambiguous since it admits two solutions with the the following values for parameters:

\[
\begin{align*}
1: & \quad a = -0.0790; \quad b = 0.441; \quad c = -0.0108; \quad d = -0.692; \quad f = -0.209 \\
2: & \quad a = 0.0957; \quad b = 0.441; \quad c = -0.0923; \quad d = 0.122; \quad f = -0.0245
\end{align*}
\] (20)

The first solution does not look very promising from a physical standpoint because it has unstable processes in both the forward path and the feedback path, with the negative feedback (note that \(d\) is negative) stabilizing the closed loop. In the second solution, the feedback is positive but the forward path is stable, with a time constant of 10.5 years, and only the feedback process is unstable with a time constant of 40.8 years. Whether either of these decompositions has any physical interpretation that makes sense in the context of the climate system has not been considered yet but it is a possible topic of future research.

Another possibility is that the quasi-cyclic behaviour is the product of a closed feedback relationship between \(x_2(k)\) and \(u_2(k)\), where \(u_2(k)\) is not considered as an external input but as a product of the dynamic interaction between the two variables. Granger causality analysis does not rule out the possibility of such a closed loop relationship, particularly if such a relationship can be justified in physical terms as a ‘tele-connection’. However, initial attempts at such modelling, using state-space and vector autoregressive models, have not produced very encouraging results, again because the relationships are probably time varying or nonlinear.

6 An Alternative Two-Input Model

The two-input DBM model (18) described in the previous section was estimated using a backfitting algorithm so that it could be compared directly with the forecasting model (13) developed in section 5.1. However, from a statistical standpoint, it is preferable to estimate the model directly using the `rivcbijd` and `rivcbj` routines in the CAPTAIN Toolbox. This is because the backfitting model estimation ignores the cross coupling effects between the two TF models, while the direct estimation takes account of these. As in the previous modelling exercises, this direct identification and estimation includes an additional input to account for the initial condition effects (see previous sections and Appendix A) but the details of the TF for this second input are not reported here to avoid any confusion.

41
The verbatim results from rivcbjid are given below, where it is clear that, on the basis of the YIC and BIC identification criteria, the [1 2 2 0 0] model is best identified. Other, higher order models explain the data marginally better but not sufficiently to justify the increased number of parameters.

<table>
<thead>
<tr>
<th>den</th>
<th>num</th>
<th>del</th>
<th>AR</th>
<th>MA</th>
<th>YIC</th>
<th>Rt2</th>
<th>BIC</th>
<th>S2</th>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0.911737</td>
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</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>-0.3266</td>
<td>0.914634</td>
<td>-746.5620</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0.914962</td>
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<tr>
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<td>0</td>
<td>0</td>
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<td>0.914888</td>
<td>-747.0284</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>-0.0372</td>
<td>0.904010</td>
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<tr>
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<td>2</td>
<td>0</td>
<td>0</td>
<td>1.1413</td>
<td>0.917306</td>
<td>-741.4248</td>
<td>6.043e-03</td>
</tr>
</tbody>
</table>

The full rivcbj estimated model takes the following form when decomposed into its constituent parts:

\[
\begin{align*}
\text{TRF model} & \quad x_1(t) = \frac{b_{11}s + b_{12}}{s + a_1}u_1(t) \\
\text{AMO model} & \quad x_2(t) = \frac{b_{21}s + b_{22}}{s + a_1}u_2(t) \\
y(k) & = x_1(k) + x_2(k) + \frac{1}{1 + c_1 z^{-1}} e(k) ; \quad e(k) = \mathcal{N}(0, \sigma^2)
\end{align*}
\]  

where it will be noted that the constituent TFs have a common denominator \(s + a_1\). The noise model is identified as an AR(1) process and the residuals \(e(k)\) are both serially uncorrelated and uncorrelated with both inputs. The estimates of the parameters are shown below with the estimated standard errors in parentheses:

\[
\begin{align*}
\hat{a}_1 &= 0.0159(0.0131) ; \quad \hat{b}_{10} = 0.0415(0.0255) ; \quad \hat{b}_{11} = 0.0147(0.0063) ; \\
\hat{b}_{20} &= 0.3559(0.0414) ; \quad \hat{b}_{21} = -0.0099(0.0075) ; \\
c_1 &= 0.2378(0.0786) ; \quad \sigma^2 = 0.006 ;
\end{align*}
\]
A model with the constituent TFs having different denominators, including a second order TF for the AMO input, has also been evaluated using the rivcdd routine in CAPTAIN but there is little of no improvement in explanatory ability, with virtually the same $R_T^2$, so the additional parameters are not justified.

The model (21) explains the data well with $R_T^2 = 0.912$, as illustrated in Fig.24 which compares the full model output, shown as a full red line, with the measured GTA. Also shown are the contributions to this full model response arising from the total radiative forcing (red dash-dot line) and the AMO (blue line) inputs. If this model is compared with the previous model

![Graph](image)

**Figure 24:** Output response of the DBM model compared with the measured GTA, showing also the contributions to this response arising from the total radiative forcing and AMO inputs.
7 DBM Models and Emission Management

The models identified and estimated in previous sections are in an ideal form for use model-based control system design. For instance, Figure 27 is a block diagram of the HI-DBM model in a form that can be used for control system design. It will be noted that it contains an additional TF element in the form of a model between the carbon emissions, which can be considered as a control or management input, and the \( \text{CO}_2 \) component of the radiative forcing inputs, shown in the left-hand panel of Figure 28. The rivcd routine in CAPTAIN identifies a first order model between between this emissions input and the \( \text{CO}_2 \) of the following form:

\[
x_3(t) = \frac{4.67 \times 10^6 \pm 6.83 \times 10^7}{s + 0.0081 \pm 0.0038} u_3(t)
\]

\[
y(k) = x(k) + \xi_3(k)
\]
where $u_3(t)$ denotes the emissions input; $x_3(t)$ is the $CO_2$; and $\xi_3(k)$ is the noise component, modeled as an ARMA(10,1) process. The deterministic output $x_3(t)$ of this model is compared with measured $CO_2$ in the right hand panel of Figure 28 and we see that it provides a reasonable explanation of the $CO_2$, with $R_T^2 = 0.998$. The residual of the ARMA(10,1) model (one-step-ahead prediction errors) has zero mean and variance $1.04 \times 10^6$ but is not normally distributed. However, the model should still be suitable for control system design.

Given the control model in Figure 27, any method of model-based feedback control could be used to design a control system that adjusted the emissions input $u_2(t)$ so that the global temperature followed an acceptable trajectory into the future, so providing guidance about the range and magnitude of the emissions reduction that would be required to control temperature rise, under the assumption that the control model is a reasonable representation of the global temperature dynamics. An exercise of this kind carried
Figure 27: Block diagram of the HI-DBM model extended for use in emissions management system design.

Figure 28: Carbon emissions and CO$_2$ forcing.

out some years ago (Young and Jarvis, Young and Jarvis) using a simpler model identified at that time. Subsequently this approach has been used in other studies of a similar nature (Young, 2006; Jarvis et al., 2008). Figure 29 shows the results of a similar exercise carried out using the models identified
in the present paper. We see that the emissions are adjusted so that the atmospheric CO\textsubscript{2} and temperature stabilise at ??

Figure 30: Similar plot to Figure 29 but showing response with added uncontrolled disturbances,
8 The Latest data

This report was written in 2014 using the globally averaged data up to 2011 that were available at that time and based on earlier versions of the DBM model dating back to 2003 (when I first obtained forecasts suggesting the ‘levelling’). Since then more data have become available on the globally averaged temperatures from http://www.metoffice.gov.uk/hadobs/hadcrut4/data/current/download.html (HadCRUT4 time series: ensemble medians and uncertainties). Comparing these latest data with those used in the analysis of the previous sections, there has been a small revision in the level, arising mainly from the change in the reference level for the anomalies from 1951-1980 (as used here) to 1961-1990. Consequently, the new series is, on average, 0.041°C larger than the series used in previous chapters. In order to reasonably compare the forecasts with these new data, therefore, it is necessary to adjust the new data points by removing this average level difference. Figure 31 is based on these new data using the model (13), with the revised new data up to the latest full year of 2015 shown as blue diamonds. Note that, here, the red, smoother forecast is that

Figure 31: Forecasting results obtained using the model (13): one-year-ahead forecasts up to 2000 and 30-year-ahead thereafter.
generated using only the first two components of the quasi-cyclic component. These two components are better defined than the higher frequency components and so are more reliable. Although less reliable, the forecast using all the components, shown as a black line, is very good in this case and even captures the 2015 measurement. This was quite unexpected, as revealed by the climate science community comments on the latest data (see https://www.ncdc.noaa.gov/sotc/global/201513), where it is said that:

“The 2015 temperature also marks the largest margin by which an annual temperature record has been broken. Prior to this year, the largest margin occurred in 1998, when the annual temperature surpassed the record set in 1997 by 0.12°C (0.22°F). Incidentally, 1997 and 1998 were the last years in which a similarly strong El Nino was occurring. The annual temperature anomalies for 1997 and 1998 were 0.51°C (0.92°F) and 0.63°C (1.13°F), respectively, above the 20th century average, both well below the 2015 temperature departure”.

So it will be interesting to see whether or not the temperature will revert to general trend, as it did after 1998, and as it is forecast here by the DBM model.

8.0.1 Using Climate Models

Having no personal access to the large climate models used by climate scientists, it has not been possible to generate directly any forecasting results based on these. Moreover, there are a number of such models produced by scientists in different countries (Meehl et al., 2007), so the forecasts they would produce are somewhat different. In addition, climate scientists do not often talk about ‘forecasts’, as such; rather they refer to ‘predictions’ or projections’ into the future, neither of which are forecasts in the sense used by the present report. Certainly, their approach is radically different to that employed by either DBM or standard forecasting practitioners.

In this situation, it is necessary to obtain climate model projections from the various data-bases available on the internet. I am very grateful to Professor Geoff Allen in this regard because he has generously passed on information of this kind that he has collected and processed for the period 2000-2010. He informs me that the climate model forecasts were taken from CMIP5 decadal monthly outputs (tas/Amon) with global weighted averages
calculated using the reported weights. Outputs were converted from degrees Kelvin to anomalies based on 1961-90 climatology to be consistent with the latest HadCrut4 actual observations. Forecast anomalies were bias corrected based on historical errors for the appropriate step-ahead forecasts, averaged over start dates from 1961 to 1991 at five-year intervals (7 observations). These monthly bias-corrected anomalies in each year are averaged to produce annual forecast anomalies.

I have extracted three examples from Professor Allen’s data that relate best to the GTA data that I have used in the previous sub-sections. They are the annual.can1 based on the CanCM4 model with full-field initialization; annual.can2 based on the same model with no initialization of the ocean component; and annual.mpi based on the MPI-ESM-LR model with anomaly initialization. Fig.32 shows these climate model projections as black, blue and green dots, with the mean of the three plotted as a black line. For the same reason as in Fig.31, these projections have been reduced by 0.041°C so that they can be compared with the two DBM model forecasts shown as a red full line and a black dash-dot line representing, respectively, the forecasts using 2 and 13 periodic components in the DHR model of the quasi-cycle.

It is clear that, because of its inclusion of the quasi-cycle, the DBM forecast predicts the levelling after 2000, while the climate projections do not. Also, it is interesting to note that the climate model projections reveal short term fluctuations that are in sympathy with the DBM forecast that uses all 13 periodic components of the DHR model for the quasi-cycle. Unfortunately, however, these fluctuations are in the opposite direction to the short term deviations in the actual GTA data, showing how difficult it is to forecast these short-term phenomena and why it is safer to rely on the forecast that includes only the two smoother, longer term components of the quasi-cycle.

In considering the above results, note that the problem with using the very large climate models for forecasting is not necessarily that they are failing to fully represent the system but rather that their complexity introduces problems of various kinds, such as the difficulties in their optimization and initialization mentioned earlier in section 1. On the other hand, these are not problems that affect the DBM model very much since both optimization and initialization are handled automatically, making it a much more suitable basis for forecasting the globally averaged data. The fact that the climate model projections do not predict the levelling after 2000 suggests either that other inputs or mechanisms that would create dynamic behaviour such as that seen in the estimated quasi-cycle component of the DBM model are not
Figure 32: Comparison of DBM model forecasts and typical climate model projections.

present. or that they are not being activated in the model simulations. One possibility in this regard is discussed briefly in the next section.

9 Conclusions

This report has shown that low order, differential equation models are able to emulate the behaviour of large climate simulation models and, when based on global climate data, produce reasonable short-to-medium forecasts of global temperature, including a prediction of the ‘levelling’ in temperatures since the start of the new Millennium. The DBM identification and estimation procedure identifies a first order, stochastic model between the annual measures of total radiative forcing and global temperature. It also reveals the presence of a quasi-cyclic component of the temperature response, with a period of about 50 years, that is well correlated with the Atlantic Multidecadal
Oscillation (AMO). When AMO is considered as a second forcing input, the resulting two input model explains 92.3% of the temperature variations and provides a good basis for further research on the dynamic relationship between the globally averaged climate variables, as well as the planning and management of future carbon emissions, based on optimal control system design.

A Estimating Initial Condition Effects in RIVC Estimation

The best way to illustrate how initial conditions can be accounted for in RIVC estimation is to consider the differential equation model (5) in section 4. This is associated with a differential equation of the form:

\[
\frac{dx(t)}{dt} + a_1 x(t) = b_0 \frac{du_1(t)}{dt} + b_1 u_1(t)
\]  

(24)

and the Laplace transform of this model (see e.g. Kuo and Golnaraghi, 2002) yields:

\[
sx(s) + x(0) + a_1 x(s) = b_0 su_1(s) + u_1(0) + u_1(s)
\]  

(25)

so that,

\[
x(s) = \frac{b_0 s + b_1}{s + a_1} u_1(s) + \frac{x(0) + u_1(0)}{s + a_1} 1.0
\]  

(26)

As a result, it is straightforward to incorporate an allowance for initial condition effects into the RIVC algorithm by adding an additional constant input set to unity over the whole observational interval. The initial conditions are then estimated as the parameter(s) of the second TF in (26) that is associated with this additional input (here, a single parameter defined by \( x(0) + u_1(0) \)); and the output of this TF adds a response to the output that accounts for the effects of the initial conditions, so ensuring a better explanation of the data over the time it takes for the effects of the initial conditions to decay. This time can be considerable (e.g. 500 years in the case of the model (5)), so accounting for the initial condition effects can be very important. Finally, note that these are theoretical results and it may be necessary to increase the order of the numerator in the second TF to fully account for the initial condition effects: in the case of the model (12) in section 5, for instance, the best estimation results are obtained when this numerator is second order, i.e. the identified model structure is \([1 1 2 0 0 1 0]\), not \([1 1 1 0 0 1 0]\).
References


Young, P. C. and A. J. Jarvis. Data-based mechanistic modelling, the global carbon cycle and global warming.


