A Control Systems Approach to Input Estimation with Hydrological Applications

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Abstract: This paper demonstrates the feasibility of a new approach to system inversion and input signal estimation based on the exploitation of non-minimal state space feedback control system design methods that can be applied to non-minimum phase and unstable systems. The real and simulated examples demonstrate its practical utility and show that it has particular relevance in a hydrological systems context.

1. INTRODUCTION

The problem of inverting linear dynamical system has a long history in control and estimation theory. A good review of the published literature is given in Gillijns [2007] [1]. While most of this research is based on optimal estimation theory, for example by the use of a special Kalman filter to perform joint *Minimum Variance Unbiased* (MVU) state and input estimation (e.g. Gillijns and Moor [2007]), one theoretical approach to the related stable inversion problem [Antsaklis, 1978] is based on feedback control theory, but it does not seem to have been applied in practice and appears to be limited to systems with stable zeros.

The approach proposed and evaluated in the present paper, initially here for linear single input, single output systems, is also based on control concepts but exploits a *Non-Minimal State Space* (NMSS) approach to control system design and treats the input estimation as a tracking problem. The performance of the proposed method is compared with both MVU estimation and the alternative PE-UIO method proposed by Sumisławska et al. [2011], based on parity equations, since this is another approach that does not require full input-state estimation.

2. SYSTEM INVERSION AND INPUT ESTIMATION USING FEEDBACK CONTROL

The method of linear system inversion and input signal estimation proposed here assumes that a constant parameter, linear *Transfer Function* (TF) model has been identified and estimated previously on the basis of discrete-time input data $u(k)$ and output output data $y(k)$, sampled at a uniform sampling interval Δt: i.e.,

$$\begin{aligned} x(k) &= \frac{B(z^{-1})}{A(z^{-1})}u(k-\delta) \quad (i) \\ y(k) &= x(k) + \xi(k) \qquad\quad (ii) \end{aligned} \qquad (1)$$

[1] ftp://zinc.esat.kuleuven.be/SISTA/gillijns/reports/PhD_Gillijns.pdf

where z^{-1} is the backward shift operator, i.e. $z^{-r}y(k) = y(k-r)$; and δ is any pure time delay of $\delta\Delta t$ time units affecting the input-output dynamics. The system (1) can be non-minimum phase or unstable. The additive noise $\xi(k)$ is introduced to indicate that, in general, the output data on which the input estimation will be based, will be noisy, so adding to the difficulty of the input estimation. It might be advantageous to model the $\xi(k)$ as an ARMA process. However, in hydrological situations, for which this approach to input estimation has been derived initially, the ARMA model may not be appropriate and the white noise source $e(k)$ can be heteroscedastic, with the variance $\sigma^2 = \sigma^2(k)$ changing over time.

Given this model, the system inversion problem can be stated as follows:

> *Given the model (1) and a new measurement $y(k)$ of the system output, over an observation interval $N\Delta t$ of N samples, generate an estimate $\hat{u}(k), k = 1, 2, \ldots, N$, of the input $u(k)$ that caused this output behaviour.*

The proposed control systems approach to the solution of this problem is illustrated in Figure 1, where the TF $P(z^{-1})/Q(z^{-1})$ represents a controller designed to make the output, denoted by $\hat{y}(k-\tau)$, track the command input, in the form of the measured output data $y(k)$, as closely as possible. In this manner, the input control signal, denoted by $\hat{u}(k-\tau)$, will then provide an estimate of the input signal, the quality of which will depend upon the control system design and the time delay τ.

This time delay τ is introduced on $\hat{y}(k-\tau)$ and $\hat{u}(k-\tau)$ in order to recognise that the control system will not respond instantaneously to the changes in $y(k)$, even if it is designed to have a very rapid, dead-beat response. Consequently, the best estimate of the input will be lagged by a time delay that is dependent on the control system design. As we shall see in the later examples, however, if the control system design procedure is powerful enough

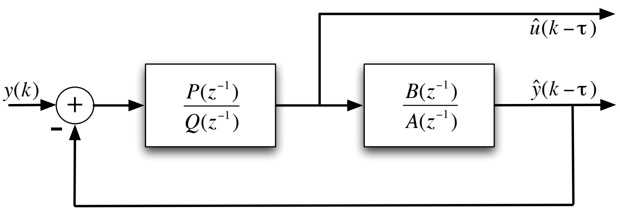

Fig. 1. PIP Control system block diagram formulated for input signal estimation.

and inherently stable, with sufficiently high gains to ensure rapid closed loop response, then the *shape* of the controlled output $\hat{y}(k-\tau)$ should closely resemble $y(k)$ as it was τ samples previously. It is then possible to adjust both the output $\hat{y}(k)$ and the control input $\hat{u}(k)$ by a simple temporal shift of τ samples to compensate for the time delay, with $\hat{u}(k-\tau)$ then providing the required estimate of the input signal $u(k)$.

Of course, the success of this approach depends critically in the performance of the control system design procedure. The next section outlines the procedure used in the later real and simulation examples: namely, high gain *Proportional-Integral-Plus* (PIP) design that exploits a *Non-Minimal State-Space* (NMSS) formulation of the control problem.

3. PIP CONTROL SYSTEM DESIGN

The NMSS approach to state variable feedback control has a fairly long history. For example, Young and Willems [1972] suggested extending the minimal state space to include an integral-of-error state variable, the feedback of which induces Type 1 servomechanism performance (zero steady state error to step commands) in the resulting state variable feedback control system. This idea was then taken further by Young et al. [1987] who showed how the natural NMSS description of the linear, discrete-time TF model

$$y(k) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} u(k) \tag{2}$$

has a state vector $\mathbf{x}(k)$ composed of the output $y(k)$ and its n past sampled values, together with the $m-1$ past sampled values of the input $u(k)$ and an integral-of-error state variable $z(k)$, i.e.

$$\begin{aligned} \mathbf{x}(k) &= [y(k) \;\; y(k-1) \;\; \ldots \;\; y(k-n+1) \\ &\quad u(k-1) \;\; u(k-2) \;\; \ldots \;\; u(k-m+1) \;\; z(k)]^T \end{aligned} \tag{3}$$

where,

$$z(k) = z(k-1) + \{y_d(k) - y(k)\} \tag{4}$$

and $y_d(k)$ is the command input into the closed loop (in the present context, the measured output $y(k)$). It is easy to see that Type 1 servomechanism performance is guaranteed by the introduction of $z(k)$: provided the closed loop remains stable and the command input $y_d(k)$ is constant, then $z(k) = z(k-1)$ in the steady state and, from (4), $y(k) = y_d(k)$. Controllability and stability are assured provided only that the polynomials $A(z^{-1})$ and $B(z^{-1})$ are coprime and that the sum of the $B(z^{-1})$ polynomial coefficients is not zero [Young et al., 1987].

The obvious advantage of this NMSS description is that all of the NMSS state variables are available for measurement and can be used in an associated PIP control system to provide for full *State Variable Feedback* (SVF) control, with all its advantages in terms of pole assignment and *Linear-Quadratic* (LQ) optimal control: i.e.,

$$u(k) = -\mathbf{g}\mathbf{x}(k); \tag{5}$$

where $\mathbf{g}$ is the $n+m$ dimensional SVF control gain vector,

$$\mathbf{g} = [f_0 \quad f_1 \quad \cdots \quad f_{n-1} \quad g_1 \quad \cdots \quad g_{m-1} \quad -k_I].$$

Provided there is no model mismatch, which is not a problem in the present context, closed loop stability is maintained by this PIP control law and very rapid closed response can be engendered by specifying high gain, dead beat or near dead beat response characteristics.

For instance, in the PIP-LQ design utilized in the later examples, the cost function takes the usual form:

$$J = \sum_{k=0}^{\infty} \{\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + ru^2(k)\} \tag{6}$$

where the positive definite weighting matrix $\mathbf{Q}$ is most often defined as a diagonal matrix, i.e.

$$\mathbf{Q} = diag\,(\,q_y \;\cdots\; q_y \;\, q_u \;\cdots\; q_u \;\, q_e\,)\,. \tag{7}$$

Moreover, because of the special nature of the NMSS formulation, r in (6) is typically set equal to q_u. Using this approach, q_y, q_u and q_e, which represent the partial weightings on the output, input and integral-of-error variables in the NMSS vector $\mathbf{x}(k)$, are usually defined as follows:

$$q_y = \frac{W_y}{n} \quad ; \quad q_u = \frac{W_u}{m} \quad ; \quad q_e = W_e \tag{8}$$

so that the total weightings are given by W_y, W_u and W_e. In relation to command input tracking, which is of greatest importance in the present context, it is the weighting W_e that most controls performance, so q_e is optimized, with q_y and q_u maintained at unity.

Normally, rapid closed loop response, of the kind required in the present application, is undesirable because it can lead to signal saturation and nonlinear dynamics in practical applications. In the present context, however, this is of no concern since the control system is only being used analytically to generate the control signal $\hat{u}(k-\tau)$ that allows $\hat{y}(k-\tau)$ to track $y(k)$ and so provide an estimate of the unknown input signal.

Full details of NMSS-based discrete-time PIP control are given in the above reference, while Taylor et al. [2000], together with the previous references cited therein, show how the methodology can be generalized and applied to continuous-time and δ-operator models, time delay systems and as a basis for *Linear Exponential-of-Quadratic* (LEQ) robust control system design. More recent publications (e.g. [Wang and Young, 2006, Exadaktylos et al., 2009]) show how the same NMSS formulation can be used for model predictive control.

4. THE INPUT ESTIMATION (INPEST) ALGORITHM

Given a PIP-LQ controller formulated in the manner of Figure 1, the *INPput ESTimation* (INPEST) estimation procedure can be considered as one of jointly optimizing the PIP control system weight q_e on the integral of error state, as defined in equation (8), and the *a priori* unknown τ sample time delay affecting $\hat{y}(k-\tau)$, so that the error between $\hat{y}(k-\tau)$ and the measured output $y(k)$ is as small as possible in some sense. One potential problem with this formulation is that when $\hat{y}(k-\tau)$ is generated in this manner, it may 'over-fit' $y(k)$, so tending to amplify the closed loop additive noise effects

$$\eta(k) = G_{CL}\xi(k), \tag{9}$$

where G_{CL} is the closed loop TF, i.e.,

$$G_{CL} = \frac{P(z^{-1})B(z^{-1})}{P(z^{-1})B(z^{-1}) + Q(z^{-1})A(z^{-1})}. \tag{10}$$

This, in turn, yields an insufficiently smooth estimate $\hat{u}(k)$ of the unknown input $u(k)$.

In order to avoid such over-fitting, it may be necessary to impose some constraints on the optimization so that the solution is consistent with the control input $\hat{u}(k)$ satisfying certain requirements based, for instance, on our previous knowledge of the system and the likely nature of the unknown input. For instance, it could be fairly smooth, at one extreme; or subject to abrupt changes at the other. Examples of such behaviour are considered in the following practical and simulation examples.

Based on the above considerations, one way to carry out this kind of optimization is to consider it in the following simple optimization terms:

$$\begin{aligned} \{\mathring{q}_e, \mathring{\tau}\} &= \arg\min_{q_e,\tau} \boldsymbol{\mathcal{J}}(q_e,\tau) \\ \boldsymbol{\mathcal{J}}(q_e,\tau) &= \sum_{k=\tau+1}^{N} \epsilon^2(k) + \lambda\{\nabla\hat{u}(k)\}^2 \\ \epsilon(k) = y(k) - \hat{y}(k-\tau); &\quad \nabla\hat{u}(k) = \hat{u}(k) - \hat{u}(k-1) \end{aligned} \tag{11}$$

where $\mathring{q}_e$ and $\mathring{\tau}$ are the optimum values of q_e and τ; λ is a Lagrange multiplier that penalises the rate of change of the control input; $\epsilon(k)$ is the tracking error between $y(k)$ and $\hat{y}(k-\tau)$; and $\nabla\hat{u}(k)$ is the rate of change of $\hat{u}(k)$. Since this is deterministic optimization, the user then needs to assess the nature of the $\hat{u}(k)$ estimates obtained with different values of λ in relation to the explanation of $y(k)$ by $\hat{y}(k)$, noting that $y(k)$ is the sum of the actual output and the noise $\xi(k)$.

There are various possible approaches to the selection of λ, and this is the subject of on-going research. However, if the variance of the tracking error $\epsilon(k)$ is plotted for a range of λ values, then it has a 'plateau' of similar values for increasing λ, until the multiplier has a first noticeable effect on the error. This represent the lowest feedback gain q_e that is consistent with good tracking and, therefore, provides maximum smoothing provided by λ, without degrading the tracking error and the resultant input estimation. In the examples below, therefore, the value of λ is selected in this manner.

Traditional methods of off-line estimation often involve smoothing of some kind. This seems desirable in the present context in order to compensate for the high-pass nature of the inversion process. As we shall see in the subsequent sections 5 and 6, it is possible to introduce pre- or post-processing stages into the INPEST estimation procedure, based on recursive *Fixed Interval Smoothing* (FIS) smoothing: see e.g. Young [2011], Young and Pedregal [1996].

Although the INPEST method has only been considered so far in deterministic terms, the later examples demonstrate that it is robust and competes well with alternative approaches, producing estimates that have low error variance and make good physical sense. Clearly, it could be extended in various ways (see later in section 9). But such extensions are not considered in the present paper: the aim here is simply to demonstrate the feasibility and practical utility of the approach in its simplest form, particularly in the context of hydrological and other environmental systems.

5. COMPARISON WITH OTHER METHODS

Sumisławska et al. [2011] describe their PE-UIO method of input estimation and compare the simulation results of PE-UIO (using various parity space orders) with those produced by the MVU method of Gillijns and Moor [2007]. They use a state space model with two inputs, one of which is known. However, if this second input effect is removed, then the TF model takes the form of equation (1) with:

$$\begin{aligned} A(z^{-1}) &= 1 + 0.05z^{-1} - 0.765z^{-2} \\ B(z^{-1}) &= 1 + 0.55z^{-1} - 0.38z^{-2};\ \delta = 1 \end{aligned} \tag{12}$$

with the zero mean, serially uncorrelated, additive noise $\xi(k)$ having variance $\sigma^2 = 1.0$, yielding 6.6% noise by standard deviation. The total simulated data set comprises 20,000 samples but Fig. 2 presents the results obtained over a short 500 sample segment of these data and is similar to Fig. 3 in Sumisławska et al. [2011] [2]. However, it also includes the results obtained using the INPEST method with $\lambda = 0.0001$, and optimized parameters of $\mathring{q}_e = 0.63$ and $\mathring{\tau} = 3$. The full comparative results using these optimized parameters are presented in Table 1, which shows that INPEST and PE-UIO yield very similar error variances and both are superior to MVU, which appears to have some problems handling *Non-Minimum Phase* (NMP) systems.

Table 1. Comparative Error Variance Results

Method	$N = 500$	$N = 20,000$	$LowNoise$
PE-UIO	0.26	0.27	0.031
INPEST	0.24	0.32	0.000013
INPEST+FIS	0.18	0.26	-
MVU	1.35	1.31	0.000013

In fact, the best results are obtained by INPEST when the output $y(k)$ is objectively pre-processed by FIS estimation using the irwsmopt and irwsm routines in the CAPTAIN Toolbox (INPEST+FIS) [3]. These routines depend on the ability of the stochastic *Random Walk* (RW) or *Integrated Random Walk* (IRW) models to satisfactorily represent the dynamic behaviour of the output measurement $y(k)$. These results suggest that if, as in this example, they are able to do this, then such pre-processing could be a useful option for INPEST. If not, then the dhr or dlr routines provide for alternative FIS estimation that is capable of representing more complex stochastic behaviour (see [Young, 2011]).

Note that, although INPEST and PE-UIO perform similarly in this example, PE-UIO has difficulty with some of

[2] For full details, see Sumisławska et al. [2011] but note that Fig. 3 has some errors so cannot be compared directly with Fig. 2 here.
[3] see `http://captaintoolbox.co.uk/Captain_Toolbox.html`

the examples discussed in subsequent sections, which are typical of 'stiff' environmental models (models with widely spaced eigenvalues), where the transfer function has a zero close to unity. Also, because it contains an approximation (due to the parity space FIR filter), it does not perform as well as INPEST or MVU when there are very low levels of noise, as we see in the final column of Table 1.

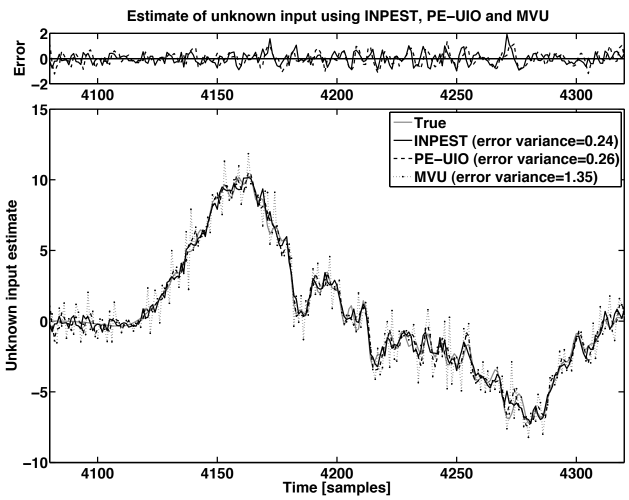

Fig. 2. Comparison of the INPEST, PE-UIO and MVU estimation results for simulated data with the estimation errors for INPEST and PE-UIO shown above.

6. EXAMPLE 2: ESTIMATION OF THE INPUT IN A TRACER EXPERIMENT ON A RIVER

This second example is based on the data plotted in Fig. 3, which shows the results of a potassium bromide (KBr) tracer experiment carried out in a wetland area as part of a study by Chris Martinez and William R. Wise of the Environmental Engineering Sciences Department, University of Florida for the City of Orlando [Martinez and Wise, 2003]. Although both the two hourly sampled input and output signals are shown here, only the downstream output $y(k)$ and a previously estimated TF model of the data will be used to estimate the upstream input $u(k)$. This input is considered as unknown, except that it is is assumed to be relatively smooth, based on normal experience with such experimental tracer data.

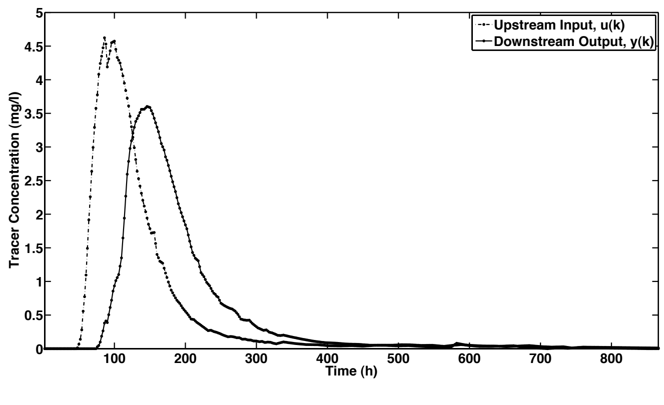

Fig. 3. Tracer experiment data used in Example 1.

A continuous-time model was estimated initially using the `rivcbj` routine in the CAPTAIN Toolbox for Matlab and this was converted to the following discrete-time model using the `c2d` routine in Matlab:

$$y(k) = \frac{0.0176 + 0.0586z^{-1} - 0.0737z^{-2}}{1 - 1.8680z^{-1} + 0.8706z^{-2}} u(k) + \xi(k) \quad (13)$$

where $\xi(k)$ is heteroscedastic noise. This model has two widely separated real roots with associated time constants of 83.7 and 17.4 hours and NMP characteristics introduced by the conversion to discrete-time. It is clear, therefore, that the TF is not invertible and that naïve inversion will fail catastrophically.

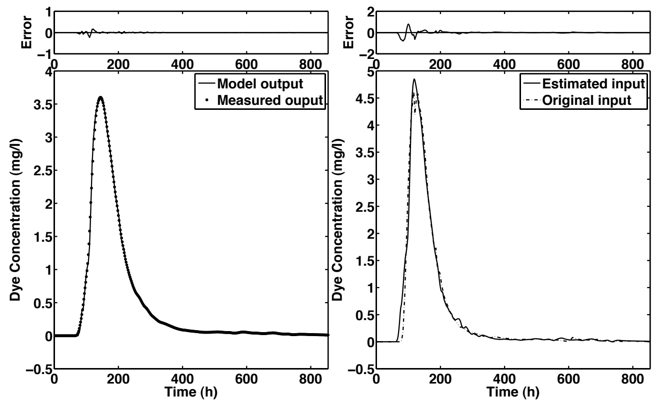

Fig. 4. Tracer experiment input estimation based on ADZ estimated model and the measured noisy input.

The right panel of Fig. 4 compares the input estimate $\hat{u}(k)$ obtained in this case with the actual input $u(k)$; while the left panel compares the resulting model output $\hat{y}(k)$ with the measured output $y(k)$. These estimates are based on optimized values of $\mathring{\tau} = 7$ and $\mathring{q}_e = 0.8$, as obtained with $\lambda = 0.001$ in the cost function $\boldsymbol{\mathcal{J}}(q_e, \tau)$. The estimate is quite good and the only obvious deficiency is at the time where the tracer concentration begins to rise. However, this is due mostly to the approximation introduced by the lumped parameter modelling of the distributed system response and not by any deficiency in the inversion procedure.

Although these results are quite good, they do not reveal too much about the nature of the estimation, which is better illustrated by simulation analysis based on the model (13). First, if $\lambda = 0$ and there is no noise on the data, then $\mathring{\tau} = 3$, $\mathring{q}_e = 407$ and the estimate $\hat{u}(k)$ is virtually perfect, even though, as expected, the naïve inversion estimate is grossly unstable. This demonstrates well the efficacy of this method in the low-noise situation.

Fig. 5 shows the results obtained when there is additive coloured noise generated by a first order AR model with lag coefficient 0.4 and white noise input variance $\sigma^2 = 0.002$, yielding about 6% of coloured noise by standard deviation. The estimates shown in Fig. 5 are based on optimized values of $\mathring{\tau} = 7$ and $\mathring{q}_e = 0.8$, as obtained with $\lambda = 0.0004$ in the cost function $\boldsymbol{\mathcal{J}}(q_e, \tau)$. In this case, the forward-path controller is computed as:

$$\frac{P(z^{-1})}{Q(z^{-1})} = \frac{0.757 - 1.414z^{-1} + 0.659z^{-2}}{1 - 2.5564z^{-1} + 2.123z^{-2} - 0.567z^{-3}} \quad (14)$$

and the estimate $\hat{u}(k - \tau)$ is generated by:

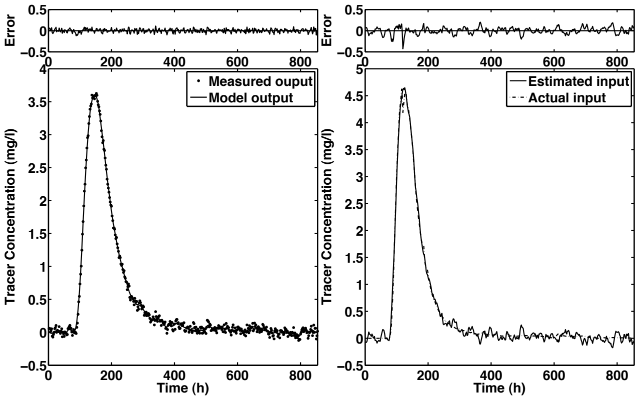

Fig. 5. Tracer experiment input estimation based on simulated data with higher level output coloured noise.

$$\hat{u}(k-\tau) = \frac{0.757 - 1.414z^{-1} + 0.659z^{-2}}{1 - 2.543z^{-1} + 2.168z^{-2} - 0.623z^{-3}} y(k) \tag{15}$$

A measure of the uncertainty in the input estimate can be obtained by recourse to simple *Monte Carlo Simulation* (MCS) analysis. Fig. 6 shows the results of such analysis for 100 realizations, demonstrating that the estimate is unbiased and its variance is acceptable.

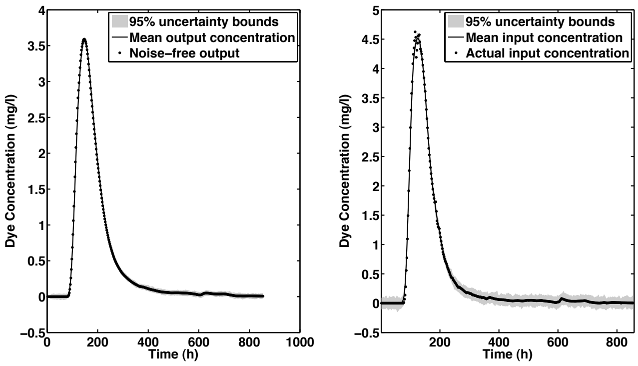

Fig. 6. Tracer experiment input estimation based on simulated data with higher level output coloured noise: MCS simulation results.

7. EXAMPLE 3: MORE DIFFICULT SITUATIONS

The input-output data in the tracer experiment example are quite smooth with no overt non-minimum phase characteristics, so it is interesting to consider some potentially more difficult situations.

First, Fig. 7 shows the results obtained when the input is changed to a unit step. Here, $\lambda = 0$, the optimized values are $\mathring{q}_e = 1000$ and $\mathring{\tau} = 3$ and the initial input estimate detects the step change well but is quite noisy, as one would expect. However, the post-optimization FIS smoothing, using the irwsm routine with NVR=0 and a single intervention point at the detected step change location (after 466 hours), removes noise and produces a near-perfect estimate of the unit step input.

Fig. 8 shows results obtained when the tracer simulation model is modified to have strong, visible NMP characteristics, with a clearly defined initial negative response

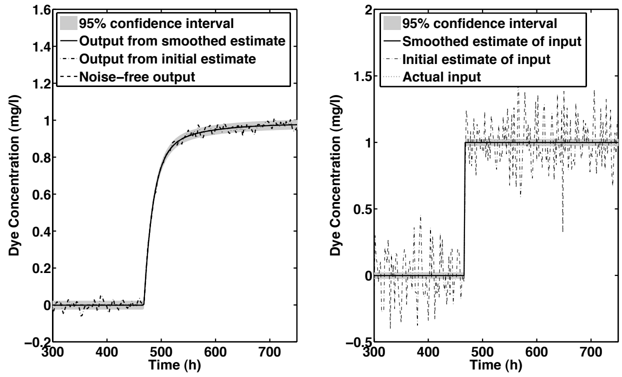

Fig. 7. Input estimation for a step input.

to the input. The problem in this situation is that the inherent NMP response characteristics are carried through to the closed loop response, leaving an anomalous negative response in the estimate.

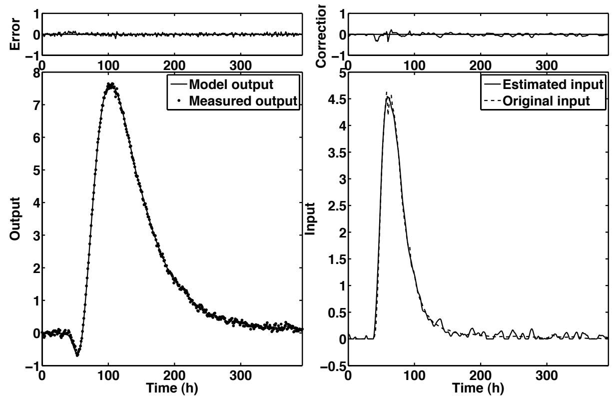

Fig. 8. Input estimation for a non-minimum phase system.

Fig. 8 illustrates a solution to this problem that utilizes the 'command input anticipation' version, PIP-COM, of the PIP design method that is able to remove NMP features from the closed loop response [Taylor et al., 2000]. The results were obtained with an optimized gain parameter of $\mathring{q}_e = 1.0$, as obtained with $\lambda = 0.0001$. The shift $\tau = 40$ is not optimized, since it is defined by the PIP-COM design.

8. EXAMPLE 4: ESTIMATION OF EFFECTIVE RAINFALL FROM MEASURED FLOW IN A RIVER

This example is a little different from the previous ones since the objective is not to estimate the input but to diagnose where the known input is unable to satisfactorily explain the measured output and to suggest how it needs to be modified to correct for such errors. The input in question is the effective rainfall in a *Data-Based Mechanistic* (DBM) rainfall-flow model relating effective rainfall to flow in a river, where the heteroscedastic noise variance changes as a function of the flow [see e.g. Young, 2001].

Various models have been suggested for such effective rainfall generation (see e.g. Beven [2001]) but none of these is able to explain the flow very accurately. Consequently, the identification of the magnitude and location of significant errors in the generated effective rainfall series could help to further research in this area of study. Fig. 9 shows the MCS results for a typical example of such analysis.

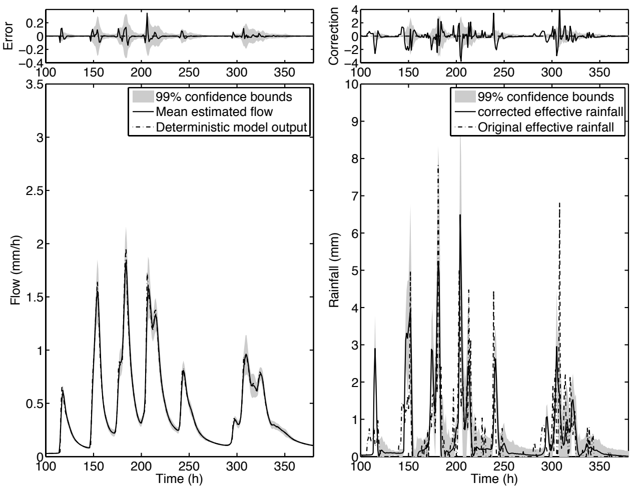

Fig. 9. Estimation and correction of effective rainfall.

In this example, it is known that the input effective rainfall has significant errors and, being a rainfall measure, the estimated input $\hat{u}(k) \geq 0, \ \forall k$. Fig. 9 shows the results obtained with $\lambda = 0.00001$, which results in optimized parameters of $\mathring{q}_e = 67$ with $\mathring{\tau} = 3$. The coefficient of determination associated with the simulated flow output is $R_T^2 = 0.993$ (compared with $R_T^2 = 0.897$ for the original DBM model).

Closer evaluation of the results shown in Fig. 9 suggests that the statistically significant adjustments to the effective rainfall, as suggested by those corrections that are larger than the confidence bounds in the upper 'Correction' graph of the right hand panel in the figure, make good physical sense: e.g. some errors appear due to changes in the pure time delay that are not present in the model.

9. CONCLUSIONS

This paper has demonstrated the feasibility of a new approach to system inversion and input signal estimation, INPEST, that is based on the exploitation of non-minimal state space feedback control system design methods and will work with non-minimum phase and unstable systems. In the comparative study of section 5, the results show that INPEST has very similar performance to the PE-UIO method, based on parity equations. However, it has some advantages over PE-UIO when applied to 'stiff' models of the kind encountered in environmental modelling and examined in the other examples of sections 6 to 8, where the TF has zeros close to unity. From the results presented here, INPEST appears to be robust in practical application and quite easy to use: it provides a simple way of both estimating an input perturbation from output measurements and diagnosing limitations in the model and/or the measured data. It could also be used for fault detection: e.g. when faults are modelled as an unknown input [Gillijns, 2007].

In its present form, the INPEST method is deterministic and requires numerical optimization, but the favourable results obtained for both real and simulated, non-minimum phase systems, in comparison with alternative stochastic methods, suggest that it is suitable for future research on several topics. These include: formulating the method more formally in theoretical terms: for example, by considering stochastic analysis using PIP-LQG control system design; extending it to multivariable systems using multivariable PIP control system design methods; and adding an integrated method of fixed interval smoothing. In practical terms, however, this paper has demonstrated that the INPEST approach already works well enough to be of significant practical utility in the hydrological and environmental sciences, for which it has been designed primarily. Moreover, it can be implemented very simply using available CAPTAIN Toolbox routines.

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